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Attestation

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The attached documents are exact copies of the text in which the European patent application described on the following page is deemed to have been filed.

Les documents joints à la présente attestation sont conformes au texte, considéré comme initialement déposé, de la demande de brevet européen qui est spécifiée à la page suivante.

Patentanmeldung Nr. **Patent application No.** **Demande de brevet n°**

01107530.6 / EP01107530

The organization code and number of your priority application, to be used for filing abroad under the Paris Convention, is EP01107530.

Der Präsident des Europäischen Patentamts;
Im Auftrag

For the President of the European Patent Office

Le President de l'Office européen des brevets
p.o.

R.C. van Dijk

YB04104

Anmeldung Nr:
Application no.: 01107530.6
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ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)
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Bezeichnung der Erfindung / Title of the invention / Titre de l'invention:
(Falls die Bezeichnung der Erfindung nicht angegeben ist, siehe Beschreibung.
If no title is shown please refer to the description.
Si aucun titre n'est indiqué se référer à la description.)

Sampling methods, reconstruction methods, and devices for sampling and/or reconstructing signals

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PATENTS & TECHNOLOGY SURVEYS

Désignation des inventeurs

Dossier N° : EPFL-2-EP

Demande N° :

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Contrat conclu le :	1.7.1995	15.9.1997	1.3.1998
Remarques :			

*Pour un nombre d' inventeurs supérieur, merci d'utiliser un formulaire supplémentaire.

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Lieu et date

Lausanne,
26 Février 2001

Signature(s)

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Our reference EPFL-2-EP / BB
Your reference
Date March 26, 2001

FAX (168 pages)

New EP Patent Application
"Sampling methods, reconstruction methods, and devices for sampling and/or reconstructing signals"

Dear Madam/Sir

Enclosed herewith we are filing a request for a patent application in the name of ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL) for "Sampling methods, reconstruction methods, and devices for sampling and/or reconstructing signals".

Please note that the applicant does not wish for the moment to pay any fees.

Enclosed we are also sending you a "receipt for documents" which we kindly ask you to return to us duly stamped or perforated.

We look forward to your confirmation and remain

Very truly yours

Patents and Technology Surveys Ltd.

Christophe SAAM
European Patent Attorney

Encls.: a/s

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01107530.6

Sampling methods, reconstruction methods, and devices for sampling and/or reconstructing signals.

The present invention relates to sampling methods, related reconstruction methods, and devices for sampling and/or reconstructing signals.

In many communication systems, the analog waveform is digitized as soon as possible, so that all further processing can be done in the discrete-time digital domain.

In the prior art, such digitization requires sampling the continuous-time analog waveform at some regular interval, called sampling period T. Its dual, $\omega_s=2\pi/T$, is called the sampling frequency. Following the Shannon theorem, the signal to be sampled is band-limited to $\omega_m = \omega_s/2$ with a band-limited filter. The band-limitation introduces a distortion of the signal.

In other cases, a higher sampling frequency ω_s is used in order to satisfy the Shannon theorem without band-limiting the signal to be sampled. A high sampling frequency however requires fast, expensive and power-consuming A/D converters, fast digital circuits and a waste of storage place for storing the digitized signal.

An object of the invention is to find a sampling scheme for sampling at least some classes of non-band-limited signals, and for allowing an exact reconstruction of those signals.

An other object of the invention is to find an improved method for sampling at least some classes of band-limited signals with a sampling frequency lower than the frequency given by the Shannon theorem, and still allowing an exact reconstruction of those signals.

An other object of the invention is to find an improved method for sampling signals which, even if they may be band-limited, can only be

seen through an imperfect measuring device having its own transfer characteristic.

An other object of the invention is to find an improved method for sampling signals that uses uniformly spaced sampling times, and thus that is easier to implement than methods using non-uniform sample intervals.

According to the invention, those technical problems are solved with a new sampling method which allows sampling of a whole newly defined class of signals, including some non-band-limited signals, at a lower sampling rate than in the prior art.

More specifically, those problems are solved with a new sampling method comprising the step of convoluting said first signal ($x(t)$) with a sampling kernel ($\phi(t)$) and a regular sampling frequency (f), said sampling kernel and said sampling frequency being chosen such that the sampled signal ($y_s[n]$) is a complete representation of said first signal ($x(t)$), allowing a perfect reconstruction of said first signal, wherein the sampling frequency (f) is lower than the Shannon frequency given by the Shannon theorem, but greater than or equal to the innovation rate of said first signal ($x(t)$).

According to an other feature of the invention, the new method allows for a perfect reconstruction of the sampled signal. The reconstruction method only needs knowledge about the class of the signal to reconstruct (for example a periodical stream of Dirac pulses, a piecewise polynomial signal, etc); it does not require any information on the signal itself, such as e.g. the times of the pulses or the position of the pieces in a piecewise polynomial signal.

Applications of the inventive sampling and reconstruction methods can be found in many technical fields, including signal processing, communications systems and biological systems. The sampling and reconstruction method can be used for sampling and reconstructing wide-band communication signals, such as CDMA signals, and ultra-wide band

communication signals, such as pulse-position modulated signal and time-modulated ultra-wide band signals, among others.

The invention starts from a definition of a new class of signals which is broader than the usual class of band-limited signals. More specifically, the invention concerns signals with a finite rate of innovation ρ, i.e. signals having an (at least locally) finite number of degrees of freedom K per unit of time T. The sampling methods of the invention is based on the surprising finding that periodic or finite-length signals in this newly defined class of signals can be exactly specified by K samples.

Various examples of sampling methods, adapted to different classes of signals and using various sampling kernels, are described in the following papers. In each case an exact reconstruction method which needs to solve only structured linear systems is described. The one skilled in the art will understand however that the invention is not limited to the specific examples given in this description, and that advantageous technical effects can be obtained by sampling other kinds of signals with a finite rate of innovation.

Signals with a finite rate of innovation are very common. They include for example most signals produced by numerical circuits, D/A converters and processors, where the rate of innovation is limited by the clock rate of the electronic. They also include numerically produced audio and video signals.

The invention will be better understood with the examples given in the following sections:

The first section (slides) describes briefly with the illustration of block-diagrams the sampling and reconstruction methods used for various classes of signal with a finite rate of innovation.

The second section (Sampling Signals with finite rate of innovation) gives a more formal definition of the class of signals with a

finite rate of innovation or with a locally finite rate of innovation. It explains in more detail various sampling methods and reconstruction methods for signals with a finite rate of innovation.

The third section (Sampling periodic signals with finite rate of innovation) describes sampling methods and reconstruction methods used for periodic signals with a finite rate of innovation.

The fourth section (Sampling signals with finite rate and finite local rate of innovation) describes sampling methods and reconstruction methods used for finite length signals with a finite rate of innovation, and for signals with a finite local rate of innovation.

The fifth section (Sampling of Piecewise Polynomial Signals) extends the methods to bidimensional signals, among others.

The sixth and last section (Appendix 2.A) describes the annihilating filter methods which are used in various signal reconstruction methods.

The sampling and reconstruction method implies some calculations which can be done by a dedicated electronic circuit or, more conveniently, by a general-purpose processor or by a digital signal processor. The invention can be marketed as a product such as a dedicated circuit, a programmed circuit or a software product containing a program enabling a processing circuit to carry out the inventive method when said program is executed.

Claims

1. Method for sampling a first signal (x(t)) having a finite rate of innovation (ρ),

said method comprising convoluting said first signal (x(t)) with a sampling kernel (ϕ (t)) and using a regular sampling frequency (f),

said sampling kernel and said sampling frequency being chosen such that the sampled signal ($y_s[n]$) is a complete representation of said first signal (x(t)), allowing a perfect reconstruction of said first signal,

characterized in that said sampling frequency (f) is lower than the frequency given by the Shannon theorem, but greater than or equal to the innovation rate of said first signal (x(t)).

2. Sampling method according to claim 1, wherein said first signal (x(t)) is not band-limited.

3. Sampling method according to claim 1 or 2, wherein said first signal (x(t)) is a discrete-time signal.

4. Sampling method according to claim 3, wherein said first signal (x(t)) is a periodic stream of weighted Dirac pulses, and wherein said sampling kernel (ϕ (t)) is a periodic sinc signal.

5. Sampling method according to claim 4, further comprising a preliminary step of derivating said stream of pulses from an input signal.

6. Sampling method according to claim 5, wherein said input signal is a periodical piecewise polynomial signal of degree R with K pieces per period, said preliminary step comprising differentiating R+1 times said input signal.

7. Sampling method according to claim 2, wherein said first signal (x(t)) is a finite stream of weighted Dirac pulses, and wherein said sampling kernel (ϕ (t)) is an infinite sinc signal.

8. Sampling method according to claim 2, wherein said first signal (x(t)) is a finite stream of weighted Dirac pulses, and wherein said sampling kernel (ϕ (t)) is a Gaussian signal.

9. Sampling method according to claim 1, wherein said first signal (x(t)) has a rate of innovation (ρ_T) which is locally finite over a window of finite width (τ).

10. Sampling method according to claim 8, wherein said first signal is a bilevel signal.

11. Sampling method according to claim 9, wherein said first signal is a CDMA signal.

12. Sampling method according to claim 1, wherein said first signal is a ultra-wide band signal such as a pulse-position modulated signal.

13. Sampling method according to claim 8, wherein said first signal is derived from a piecewise polynomial signal.

14. Sampling method according to one of the claims 9 to 13, wherein said sampling kernel (ϕ (t)) is a spline function.

15. Sampling method according to one of the claims 9 to 14, wherein said sampling kernel (ϕ (t)) is a box function.

16. Sampling method according to one of the claims 9 to 14, wherein said sampling kernel (ϕ (t)) is a hat function.

17. Sampling method according to claim 1, wherein said first signal (x(t)) is a measure of a signal with a finite rate of innovation.

18. Sampling method according to claim 1, wherein said first signal (x(t)) is a bidimensional signal.

19. Sampling device comprising means for carrying out the method of one of the preceding claims.

20. Computer program product directly loadable into the internal memory of a digital processing system and comprising software code portions for performing the methods of one of the claims 1 to 18 when said product is run by said digital processing system.

21. Method for faithfully reconstructing a first signal (x(t)) from a set of 2K samples ($y_s[n]$)),

wherein the class of said signal to reconstruct (x(t)) is known,

wherein the bandwidth $|\omega|$ of said faithful reconstructed signal x(t) is higher than $\omega_m = \pi/T$, T being the sampling interval,

wherein the rate of innovation (ρ) of said faithful reconstructed signal is finite,

characterized in that said method comprises the step of solving a structured linear system depending on said known class of signal.

22. Method according to claim 21, wherein said first signal (x(t)) is a discrete-time signal, said reconstruction method including following steps:

finding 2K spectral values of said first signal (x(t)),

using an annihilating filter method for finding said first signal (x(t)) from said spectral values.

23. Method according to claim 21, wherein said first signal (x(t)) is a periodical piecewise polynomial signal, said reconstruction method including following steps:

finding 2K spectral values of said first signal (x(t)),

using an annihilating filter method for finding a differentiated version ($x^{R+1}(t)$) of said first signal (x(t)) from said spectral values,

integrating said differentiated version to find said first signal.

24. Method according to claim 21, wherein said first signal (x(t)) is a finite stream of weighted Dirac pulses, said reconstruction method including following steps:

finding the roots of an interpolating filter to find the position of said pulses,

solving a linear system to find the weights of said pulses.

25. Method according to claim 21, wherein said first signal (x(t)) is an infinite length signal in which the rate of innovation is locally finite, said reconstruction method comprising a plurality of successive steps of reconstruction of successive intervals of said first signal.

26. Method according to one of the claims 21 to 25, wherein said first signal (x(t)) is a bidimensional signal.

27. Method according to one of the claims 21 to 26, wherein said first signal is a measure of a signal with a finite rate of innovation, said method being used for reconstructing the original signal before measure.

28. Method according to one of the claims 21 to 27, wherein said first signal is a bilevel signal.

29. Method according to one of the claims 21 to 28, wherein said first signal is a CDMA signal.

30. Circuit for reconstructing a sampled signal by carrying out the method of one of the claims 21 to 29.

31. Computer program product directly loadable into the internal memory of a digital processing system and comprising software code portions for performing the methods of one of the claims 21 to 29 when said product is run by said digital processing system.

Bandlimited signals

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- Thm.:(WKS) A signal $x(t)$ BL to $[-\omega_m, \omega_m]$ is perfectly recovered from samples taken T apart if sampling rate $2\pi/T \geq 2\omega_m$.
- Sampling scheme

$$\begin{array}{c} \sum_n \delta(t-nT) \\ \downarrow \\ x(t) \longrightarrow \otimes \longrightarrow x_s(t) = \sum_n x_s[n]\,\delta(t-nT) \end{array}$$

where $x_s[n] = x_s(nT) = < x(t), \delta(t-nT) >$

- Reconstruction scheme

$$x_s[n] \longrightarrow \boxed{\text{Reconstruction sinc filter}} \longrightarrow x_{rec}(t) = \sum_n x_s[n]\,\mathrm{sinc}(t/T - n)$$

Non-bandlimited signals

- Sampling scheme

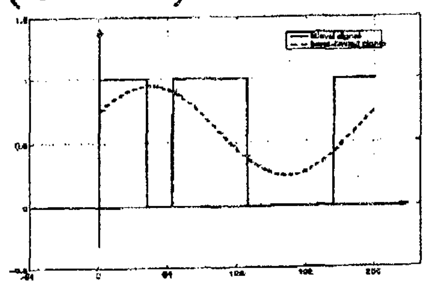
$$\sum_n \delta(t - nT)$$

$$x(t) \longrightarrow \boxed{\tilde{\varphi}(t)} \xrightarrow{y(t)} \otimes \longrightarrow y_s(t) = \sum_n y_s[n]\,\delta(t - nT)$$

where $y_s[n] = < x(t), \varphi(t - nT) >$

$\tilde{\varphi}(t) = \varphi(-t)$ is an ideal lowpass approximation filter thus $y(t)$ is bandlimited.

- Example: Bilevel signal (CDMA)



Non-bandlimited signals (2)

Example: Piecewise polynomials (Woodcut picture)

Le bestiaire ou cortège d'orphée
Guillaume Apollinaire/Raoul Dufy

Non-bandlimited signals (3)

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- Reconstruction scheme

$$y_s[n] = < x(t), \varphi(t - nT) > \longrightarrow \boxed{\begin{array}{c}\text{Reconstruction}\\ \text{Algorithm}\\ ????\end{array}} \longrightarrow x_{rec}(t)$$

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Signals with a finite rate of innovation

- Def.: Rate of innovation

$$\rho = \frac{\text{number of degrees of freedom}}{\text{unit of time}}$$

- Example 1: A bandlimited signal $[-\omega_m, \omega_m]$ with sampling interval $T = \pi/\omega_m$ is specified by 1 sample every T seconds, that is,

$$\rho = 1/T = \omega_m/\pi$$

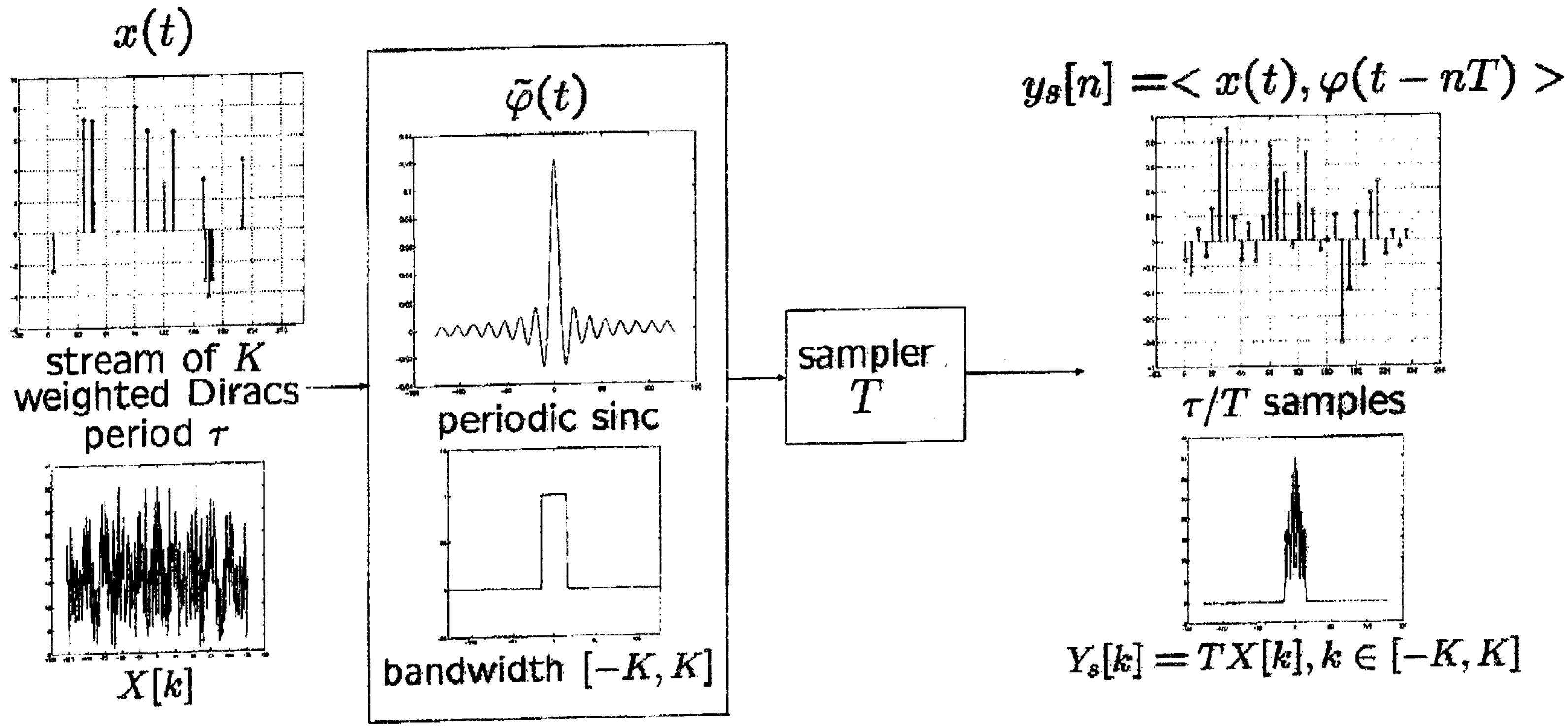
degrees of freedom per unit of time.

- Example 2: A periodic stream of K weighted Diracs with period τ is specified by K locations and K weights, that is,

$$\rho = 2K/\tau$$

Periodic stream of Diracs

- Sampling scheme



Periodic stream of Diracs (2)

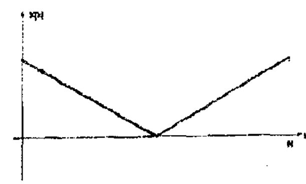
- Reconstruction scheme (Proposition 2.2)

$$y_s[n] \longrightarrow \boxed{\text{Find } 2K \text{ spectral values of } x(t)} \xrightarrow{X[k],\, k \in [-K, K]} \boxed{\text{Annihilating filter method}} \xrightarrow{t_k,\, c_k} x_{rec}(t)$$

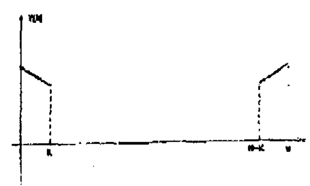
where t_k are the locations of the Diracs
and c_k are the weights of the Diracs.

Obtain $2K$ spectral values of $x(t)$

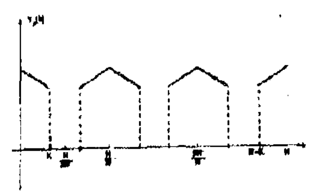
- Spectral components of stream of Diracs, $X[k]$



- Spectral components of lowpass approximation or filtered signal, $Y[k]$



- Discrete-time Fourier series of samples, $Y_s[k]$



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Annihilating filter method

- Spectral components are linear combinations of exponentials:

$$X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \underbrace{\left(e^{-i2\pi t_k/\tau}\right)^m}_{u_k^m}, \qquad m \in \mathbb{Z}.$$

- How do we find the K locations, t_k?

  A filter $1 - z^{-1}u_k$ is called an annihilating filter for u_k^m if

$$(1 - z^{-1}u_k)u_k^m \equiv 0.$$

  The filter $H(z) = \prod_{k=0}^{K-1}(1 - z^{-1}u_k)$ annihilates each exponential u_k^m

$$H(z) \cdot X(z) = 0 \Leftrightarrow \mathrm{H} * \mathrm{X} = 0.$$

  The K locations $\{t_0, t_1, \dots, t_{K-1}\}$ are given by the zeros of $H(z)$.

Annihilating filter method (2)

- How do we find $H(z)$?

  Solve the system

$$\mathbf{H} * \mathbf{X} = 0 \Leftrightarrow \sum_{k=0}^{K} H[k]X[m-k] = 0,$$

  with $H[0] = 1$

- Example: $K = 3$ and let $m = 1, 2, 3$

$$\begin{bmatrix} X[0] & X[-1] & X[-2] \\ X[1] & X[0] & X[-1] \\ X[2] & X[1] & X[0] \end{bmatrix} \cdot \begin{pmatrix} H[1] \\ H[2] \\ H[3] \end{pmatrix} = - \begin{pmatrix} X[1] \\ X[2] \\ X[3] \end{pmatrix}.$$

- Toeplitz system of K equations involving $2K$ spectral values

  $\longrightarrow$ annihilating filter coefficients $H[k], k = 1, \ldots, K$

  $\longrightarrow$ roots of $H(z)$ are $u_k \longrightarrow$ locations t_k .

Annihilating filter method (3)

- How do we find the K weights, c_k ?

  Recall

$$X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \underbrace{(e^{-i2\pi t_k/\tau})^m}_{u_k^m}, \qquad m \in \mathbb{Z}.$$

  Let $m = 0, \ldots, K-1$ and solve the Vandermonde $K \times K$ system

$$\frac{1}{\tau} \begin{bmatrix} 1 & 1 & 1 \\ u_0 & u_1 & u_2 \\ u_0^2 & u_1^2 & u_2^2 \end{bmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \end{pmatrix}$$

- Annihilating filter method $\longrightarrow u_k$ and c_k in $X[m]$.

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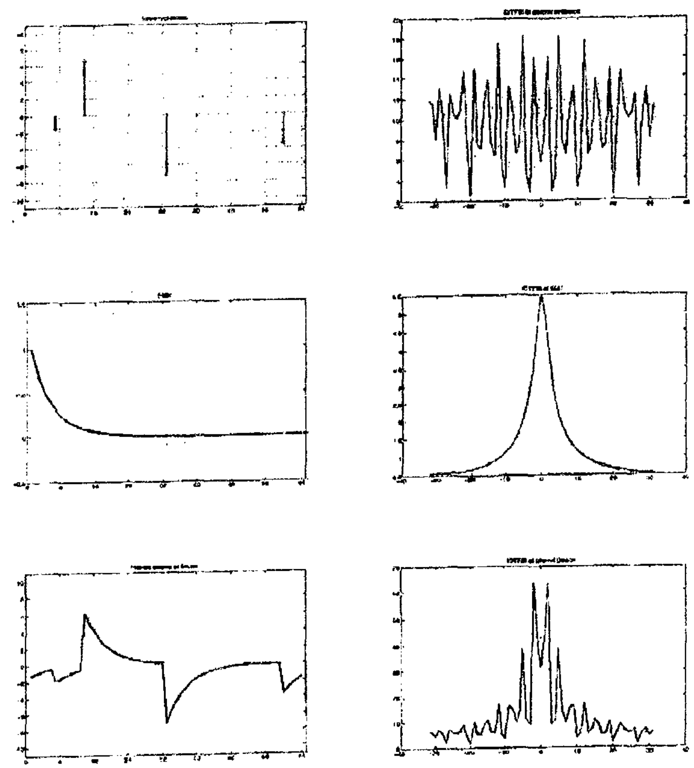
Example: filtered stream of Diracs

signal	time domain	frequency domain
Stream of K weighted Diracs, $\mathbf{x}_{Dir}$		
$*$		
Filter, $\mathbf{g}$ $G[k] \neq 0, k \in [-K, K]$		
$=$		
Filtered stream of K weighted Diracs, $\mathbf{x} = \mathbf{g} * \mathbf{x}_{Dir}$		



Filtered stream of Diracs: reconstruction algorithm

Degrees of freedom $= 2K$

1. Calculate the sample values using a sinc sampling kernel bandlimited to $[-K, K]$
2. Obtain $X[k], k \in [-K, K]$.
3. Determine the stream of Diracs from $2K$ spectral values

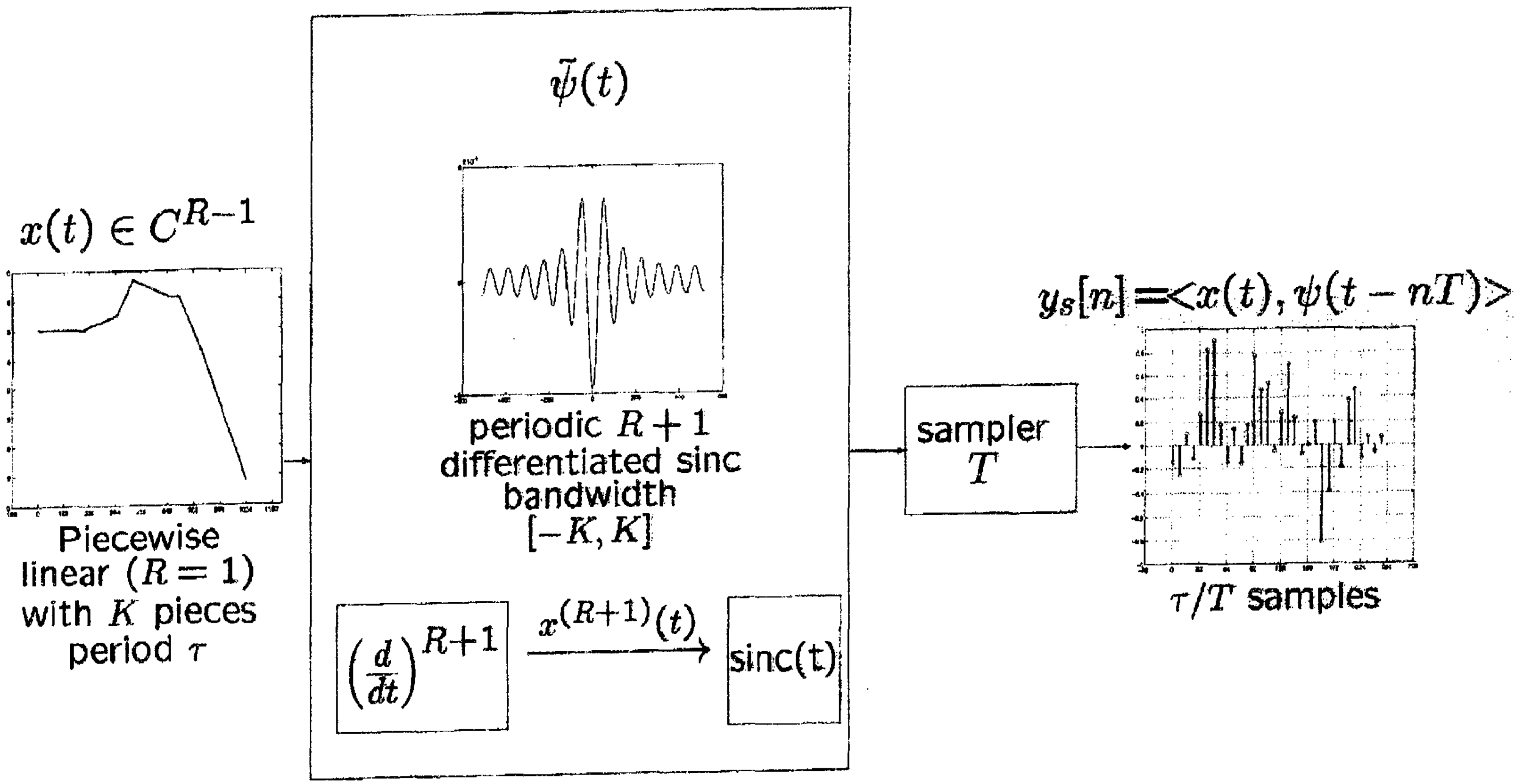
$$X_{Dir}[k] = X[k]/G[k], \quad k \in [-K, K]$$

using the annihilating filter method.

4. The signal is $\mathbf{x}_{rec} = \mathbf{x}_{Dir} * \mathbf{g}$

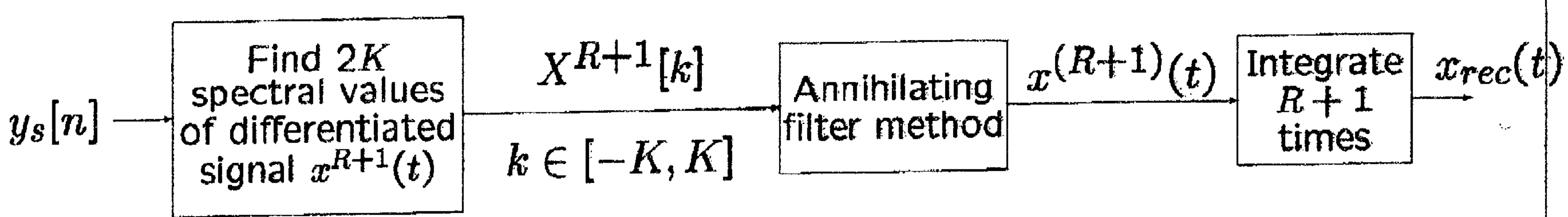
Periodic piecewise polynomial signals

- Sampling scheme

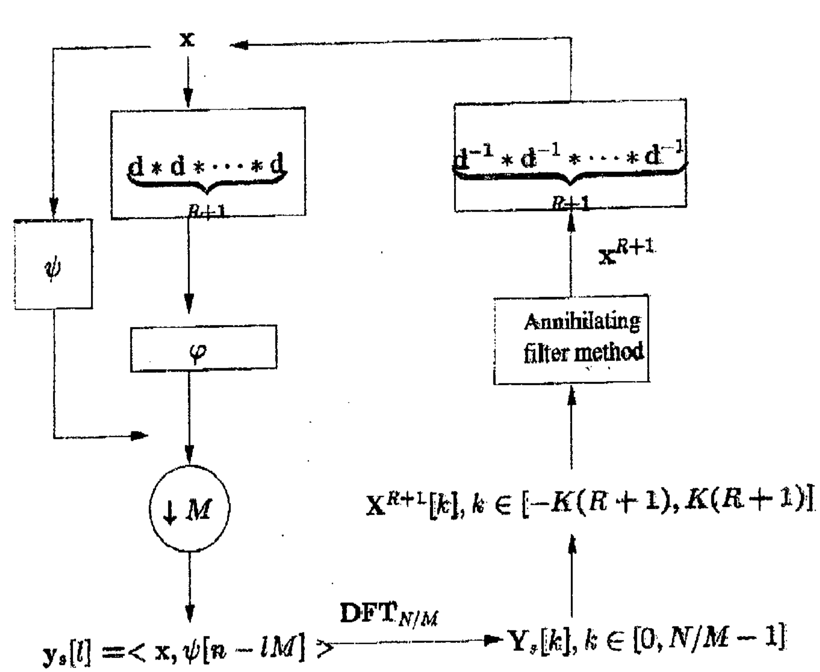


Periodic piecewise polynomial signals (2)

- Reconstruction scheme (Theorem 2.2)

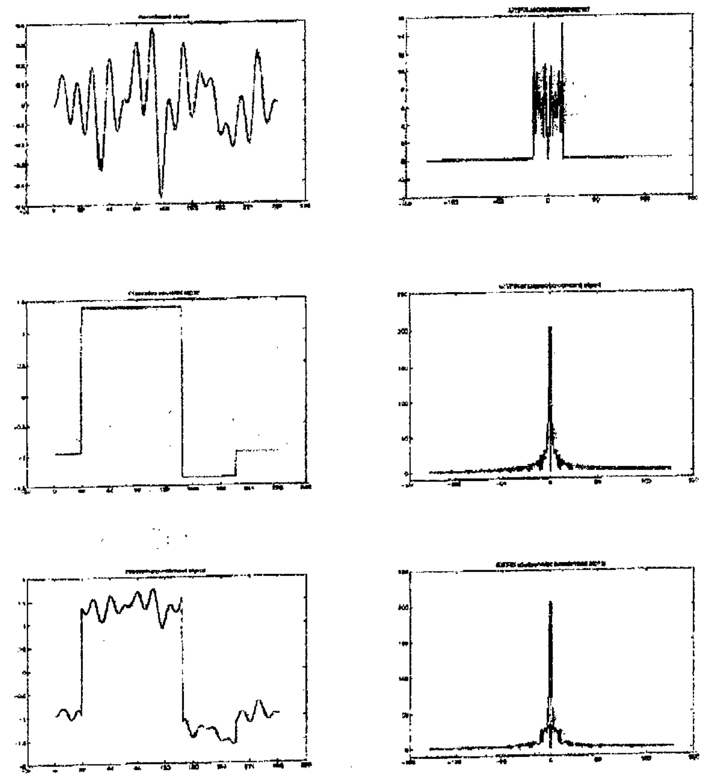


Sampling and reconstruction scheme for discrete-time periodic piecewise polynomial signals



Example: piecewise bandlimited signals

signal	time domain	frequency domain
Bandlimited, x_{BL} $X_{BL}[k] \neq 0, k \in [-L, L]$		
+		
Piecewise constant, x_{PW} $(R = 0)$ with K pieces		
=		
Piecewise bandlimited, x_{BL+PW}		



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Piecewise bandlimited: reconstruction algorithm

Degrees of freedom $= \underbrace{2K}_{\mathbf{x}_{PW}} + \underbrace{2L}_{\mathbf{x}_{BL}}$

1. Calculate the sample values using a differentiated sinc sampling kernel bandlimited to $[-(2K+L), 2K+L]$
2. Obtain $X_{PW+BL}[k], k \in [-(2K+L), 2K+L]$.
3. Determine the piecewise polynomial signal from $2K$ spectral values

$$X_{PW+BL}[k] = X_{PW}[k], \quad k \in [L+1, L+2K]$$

using the annihilating filter method.

4. The spectral values of the BL signal are given by

$$X_{PW+BL}[k] - X_{PW}[k], \quad k \in [-L, L]$$

5. The reconstructed signal $\mathbf{x}_{rec} = \mathbf{x}_{BL} + \mathbf{x}_{PW}$

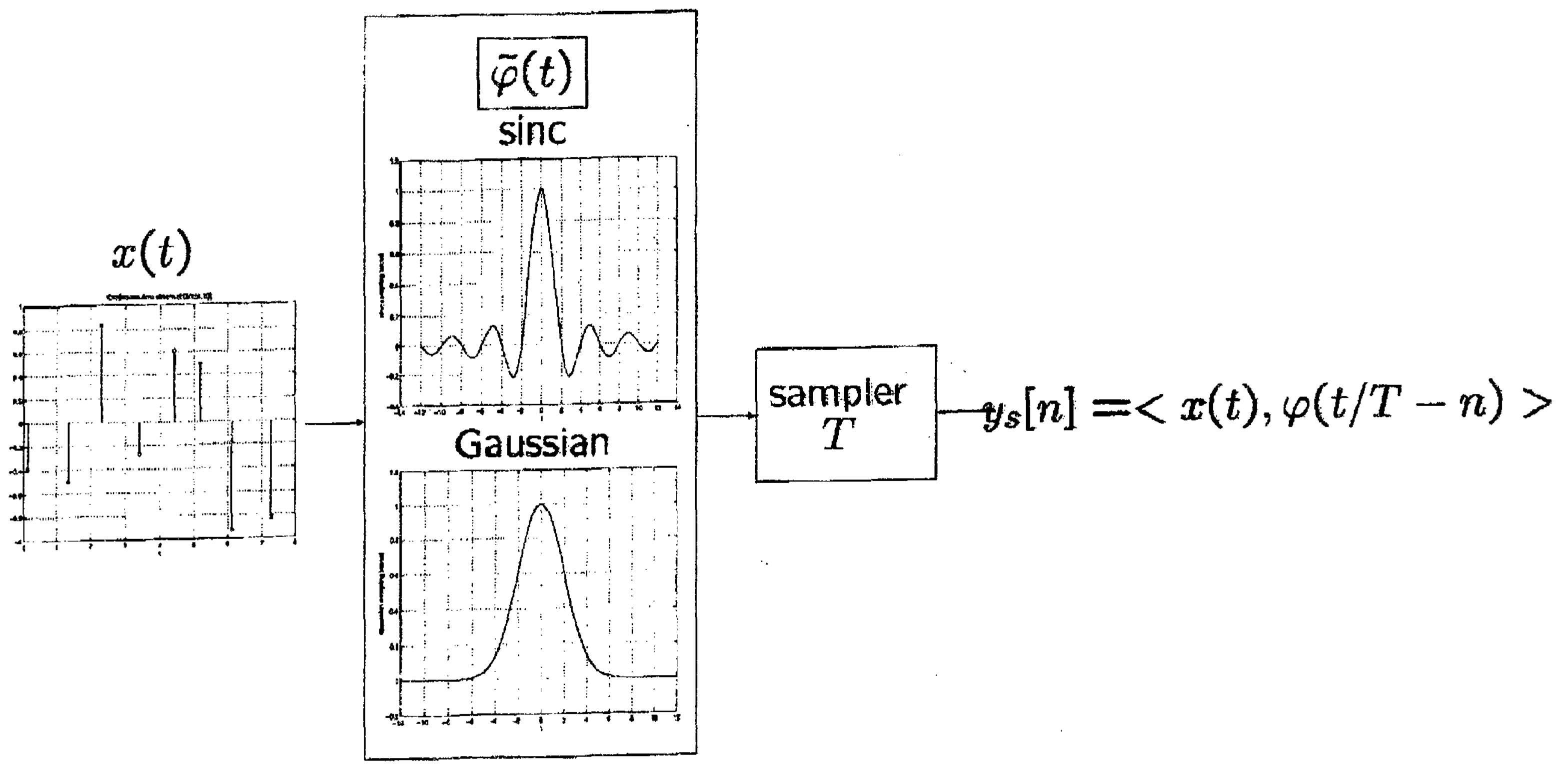
Part I : uniform sampling

Outline

- Bandlimited signals
- Non-bandlimited signals
- Signals with a finite rate of innovation
  - Periodic stream of weighted Diracs (Chapter 2)
  - Periodic piecewise polynomials (Chapter 2)
  - **Finite stream of weighted Diracs (Chapter 3)**
  - Bilevel signals with finite local rate of innovation (Chapter 3)
- Contributions

Finite stream of Diracs with infinite length sampling kernels

- Sampling scheme



Finite stream of K weighted Diracs with **sinc** sampling kernel

- Perfect reconstruction from N samples, with $N \geq 2K$, (Thm. 3.1)

$$y_s[n] = < x(t), \text{sinc}(t/T - n) > \;=\; \frac{(-1)^n}{\pi} \sum_{k=0}^{K-1} c_k \sin\left(\pi t_k/T\right) \cdot \frac{1}{\left(t_k/T - n\right)}$$

$$\Longleftrightarrow \underbrace{(-1)^n P(n)\, y_s[n]}_{Y_n} \;=\; \underbrace{\frac{1}{\pi} \sum_{k=0}^{K-1} c_k \sin\left(\pi t_k/T\right) P_k(n)}_{\text{degree } K-1} \qquad (2)$$

$P(u) = \sum_{k=0}^{K} p_k\, u^k$ with zeros at locations $t_l/T, l = 0, \ldots, K-1$

$P_k(u)$ has zeros at all locations except t_k/T

- Solve $\Delta^K Y_n = 0 \longrightarrow p_k$
- Find the roots of $P(u) \longrightarrow u_k = t_k/T \longrightarrow t_k = T u_k$
- Solve linear system (2) $\longrightarrow c_k$

Finite stream of K weighted Diracs with **Gaussian** sampling kernel

- Perfect reconstruction from N samples, with $N \geq 2K$, (Thm. 3.2)

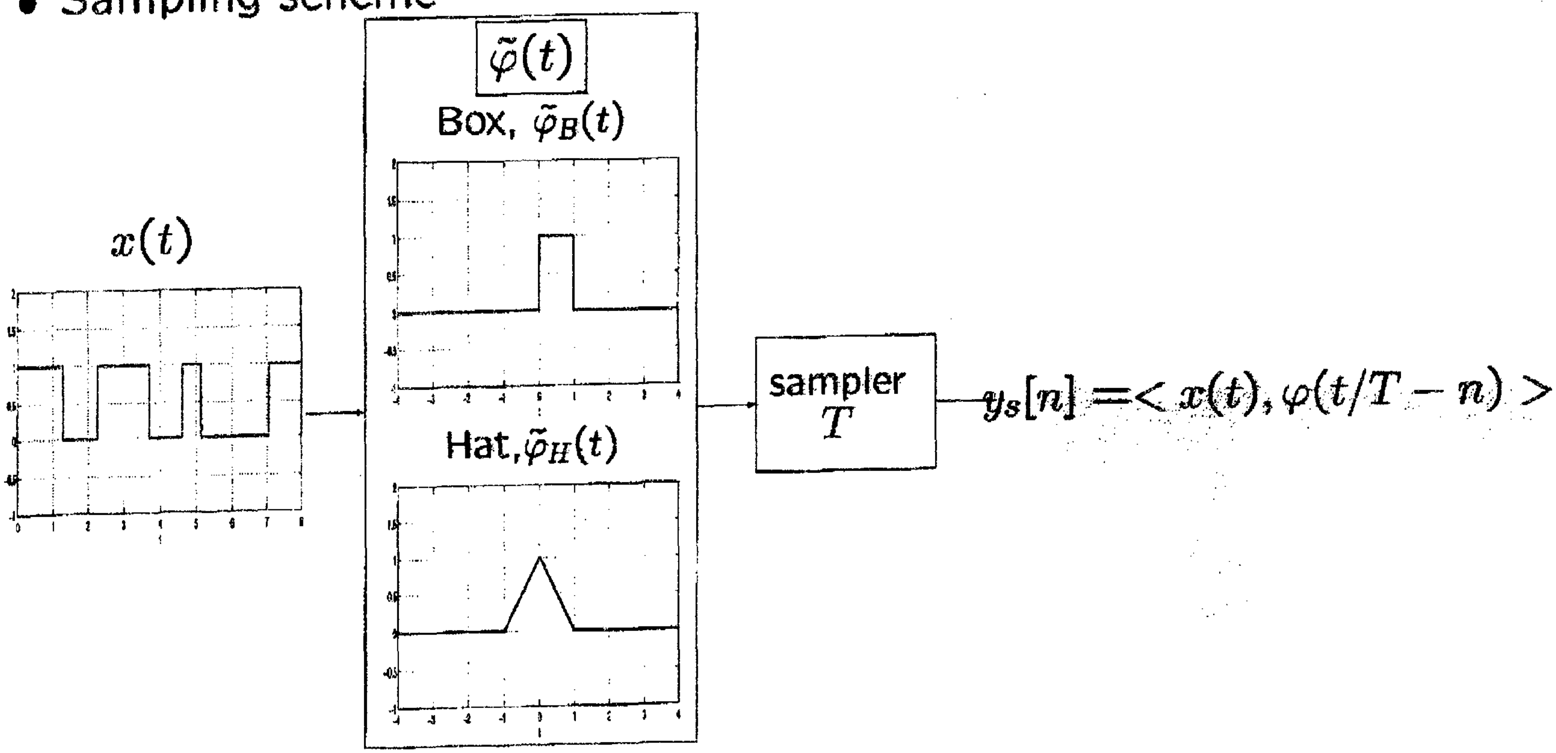
$$\begin{aligned} y_s[n] &= < x(t), e^{-(t/T-n)^2/2\sigma^2} > \\ \Longleftrightarrow \underbrace{e^{n^2/2\sigma^2} y_s[n]}_{Y_n} &= \sum_{k=0}^{K-1} \underbrace{c_k e^{-t_k^2/2\sigma^2 T^2}}_{a_k} \underbrace{e^{n t_k/\sigma^2 T}}_{u_k^n} \end{aligned} \quad (3)$$

- Y_n is linear combination of K real exponentials
  $\Rightarrow$ annihilating filter method $\longrightarrow a_k, u_k$

- From $u_k \longrightarrow t_k = \sigma^2 T \ln u_k$
- From $a_k \longrightarrow c_k = a_k\, e^{t_k^2/2\sigma^2 T^2}$

Bilevel signals with finite **local** rate of innovation

- Sampling scheme

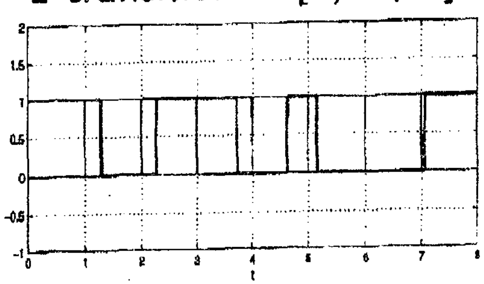


- Local reconstruction algorithms

Bilevel signal with box sampling kernel

- Box sampling kernel, $\varphi_B(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$

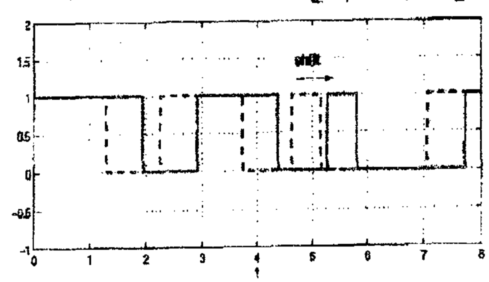
Bilevel signal with at most 1 transition in $[n, n+1]$



$$y_s[n] = \int_n^{n+1} x(t)\,dt$$
$$= \begin{cases} \int_n^{t_k} x(t)\,dt & = t_k - n \\ \int_{t_k}^{n+1} x(t)\,dt & = n + 1 - t_k \end{cases}$$

unique solution

Bilevel signal with **unknown** shift 2 transitions in $[n, n+1]$



$$y_s[n] = \int_n^{n+1} x(t)\,dt$$
$$= \begin{cases} \int_n^{t_k} x(t)\,dt + \int_{t_{k+1}}^{n+1} x(t)\,dt \\ 1 - t_{k+1} + t_k \end{cases}$$

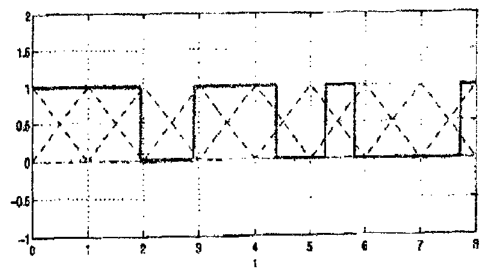
not unique solution

- Higher order splines.

Bilevel signal with hat sampling kernel

- Hat sampling kernel, $\varphi_H(t) = \begin{cases} 1-|t| & \text{if } |t| \leq 1 \\ 0 & \text{else} \end{cases}$

Bilevel signal with at most 2 transitions in $[n, n+2]$



- Solve system of quadratic equations in $[n, n+2]$

$$\begin{aligned} y_s[n] &= \int_{n-1}^{n+1} x(t)\,\varphi_H(t-n)\,dt = f_n(t_k, t_{k+1}) \\ y_s[n+1] &= f_{n+1}(t_k, t_{k+1}) \end{aligned}$$

for each configuration $(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)$ in $[n, n+2]$.

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Contributions

- Sampling theorems for discrete-time and continuous-time periodic streams of weighted Diracs and piecewise polynomial signals.

Discrete-time signals of period N	sampling rate	$\geq$	rate of innovation
Stream of K weighted Diracs	$1/M$	$\geq$	$2K/N$
Piecewise polynomial with K pieces of degree R	$1/M$	$\geq$	$2K(R+1)/N$
Continuous-time signals of period τ			
Stream of K weighted Diracs	$1/T$	$\geq$	$2K/\tau$
Piecewise polynomial $\in C^{R-1}$ with K pieces of degree R	$1/T$	$\geq$	$2K/\tau$

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Contributions (2)

- Sampling theorems for recovering a finite stream of Diracs with the infinite length **Gaussian** and **sinc** sampling kernel.
- Local reconstruction algorithms for bilevel signals with finite local rate of innovation using the **box** and the **hat** sampling kernel.

Sampling signals with finite rate of innovation

Martin Vetterli[1,2] Pina Marziliano[1] Thierry Blu[3]

Abstract

Consider the problem of sampling signals which are not bandlimited, but still have a finite number of degrees of freedom per unit of time, such as, for example, piecewise polynomials. Call the number of degrees of freedom per unit of time the rate of innovation. We demonstrate that by using an adequate sampling kernel and a sampling rate greater or equal to the rate of innovation, one can uniquely reconstruct such signals.

We thus prove theorems for classes of signals and sampling kernels that generalize the classic "bandlimited and sinc kernel" case. In particular, we show sampling theorems for periodic as well as finite length piecewise polynomials, using a bandlimited derivative kernel, as well as a Gaussian kernel. For infinite length piecewise polynomials with a finite local rate of innovation, we show exact local reconstruction using sampling with spline kernels.

All the results presented lead to computational procedures that are readily implementable, which is shown through experimental results. Applications of these new sampling results can be found in signal processing, communications systems and biological systems.

Index Terms

Sampling, generalized sampling, poisson processes, piecewise polynomials, non-bandlimited signals, analog-to-digital conversion, annihilating filters.

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CONTENTS

I. Introduction

Most continuous-time phenomenas can only be seen through sampling the continuous-time waveform, and typically, the sampling is uniform. Very often, instead of the waveform itself, one has only access to a smoothed or filtered version of it. This may be due to the physical set up of the measurement, or may be by design.

Calling $x(t)$ the original waveform, its filtered version is $x(t) * \tilde{\varphi}(t)$, where $\tilde{\varphi}(t) = \varphi(-t)$ is the convolution kernel. Then, uniform sampling with a sampling interval T leads to samples $x[n]$ given by

$$x[n] = < \varphi(t - nT), x(t) > = \int_{-\infty}^{\infty} \varphi(t - nT)\, x(t)\, dt. \quad (1)$$

This setup is shown in Fig. 1.

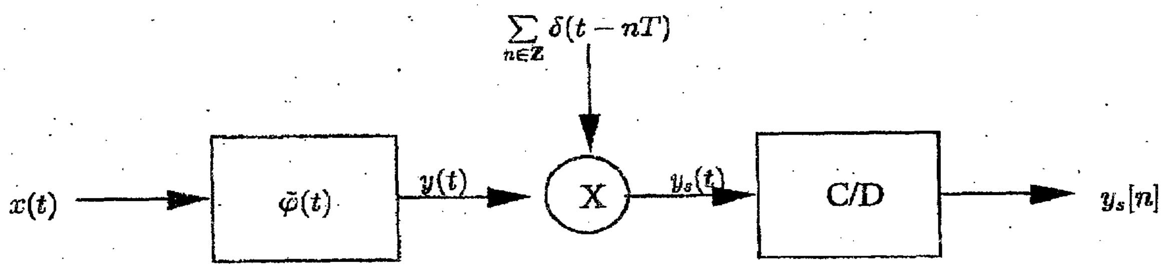


Fig. 1. Sampling set up: $x(t)$ is the continuous-time signal; $\tilde{\varphi}(t) = \varphi(-t)$ is the smoothing kernel; $y(t)$ is the filtered signal; T is the sampling interval; $y_s[n] = x[n], n \in \mathbb{Z}$ are the sample values.

When no smoothing kernel is used, we simply have $x[n] = x(nT)$, which is equivalent to (1) with $\varphi(t) = \delta(t)$. This simple model for having access to the continuous-time world is typical for acquisition devices in many areas of science and technology, including scientific measurements, medical and biological signal processing and analog-to-digital converters.

The key question is of course if the samples $x[n]$ are a faithful representation of the original signal $x(t)$. If so, how can we reconstruct $x(t)$ from $x[n]$, and if not, what approximation $\hat{x}(t)$ do we get based on the samples $x[n]$? This question is at heart of signal processing, and the

dominant result is the well-known sampling theorem of Whittaker, Kotelnikov and Shannon which states that if $x(t)$ is bandlimited, or $X(\omega) = 0, |\omega| > \omega_m$, then samples $x[n] = x(nT)$ with $T \leq \pi/\omega_m$ are sufficient to reconstruct $x(t)$ [4], [6], [9]. The reconstruction formula is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\, sinc(t/T - n), \tag{2}$$

with

$$sinc(t) = \frac{sin(\pi t)}{\pi t}. \tag{3}$$

If $x(t)$ is not bandlimited, convolution with $\tilde{\varphi}(t) = sinc(t/T)$ (an ideal lowpass filter with support $[-\pi/T, \pi/T]$) allows to apply sampling and reconstruction of $\hat{x}(t)$, the lowpass approximation of $x(t)$, or the restriction of $X(\omega)$ to the interval $[-\pi/T, \pi/T]$.

A possible interpretation of the interpolation formula (2) is the following. Any real bandlimited signal can be seen as having $1/T$ degrees of freedom per unit of time, which is the number of samples per unit of time that specify it. In the present paper, this **number of degrees of freedom per unit of time** is called the **rate of innovation** of a signal [1] , and is denoted by ρ. In the bandlimited case above, the rate of innovation is $\rho = 1/T = \omega_m/\pi$. In the sequel, we are interested in signals that have a **finite rate of innovation**, either on intervals, or on average. Take a Poisson process, which generates Diracs with independent and identically distributed (i.i.d.) interarrival times, the distribution being exponential with probability density function $\mu e^{-\mu t}$. The expected interarrival time is given by $1/\mu$. Thus, the rate of innovation is μ, since on average, μ real numbers per unit of time fully describe the process.

Given a signal with finite rate of innovation, it seems attractive to sample it with a rate of

[1] This is different from the rate used in rate-distortion theory [1]. Here, rate corresponds to a degree of freedom that are specified by real numbers. In rate-distortion, the rate corresponds to bits.

ρ samples per unit of time. We know it will work with bandlimited signals, but will it work with a larger class of signals? Thus, the natural questions to pursue are the following:

1. What classes of signals of finite rate of innovation can be sampled uniquely, especially using uniform sampling ?
2. What kernels $\varphi(t)$ allow for such sampling schemes?
3. What algorithms allow the reconstruction of the signal based on the samples?

In the present paper, we concentrate on stream of Diracs and on piecewise polynomials, which are classes for which we are able to derive sampling theorems under certain conditions. The kernels involved are related to the sinc (the bandlimited derivative), the Gaussian, and the spline kernels. The algorithms, while more complex than the standard sinc sampling of bandlimited signals, are still reasonable (structured linear systems) but also often involve root finding.

The outline of the paper is as follows. Section II formally defines signals with finite rate of innovation. Section III and Section IV consider periodic signals in discrete and continuous time respectively, and derive sampling theorems for streams of Diracs and piecewise polynomials. Both of these type of signals have a finite number of degrees of freedom, and a sampling rate that is sufficiently high to capture these degrees of freedom, together with appropriate sampling kernels, allows perfect reconstruction. Section V addresses the sampling of finite length signals having a finite number of degrees of freedom, using infinitely supported kernels like the sinc kernel and the Gaussian kernel. Again, if the critical number of samples is taken, we can derive a sampling theorem. Section VI concentrates on local reconstruction schemes. Given that the local rate of innovation is bounded, local reconstruction is possible, using for example spline kernels. Finally, Section VII derives applications of the above results, in particular to piecewise bandlimited signals, and to filtered streams of

Diracs.

In the Appendix, we introduce the "annihilating filter" method borrowed from spectral analysis. This method will be referred to in all of the proofs in Section III, Section IV as well as one in Section V-B.

II. Signals with finite rate of innovation

In the introduction, we have informally discussed the intuitive notion of signals with **finite rate of innovation**. More formally, consider functions or signals having a parametric representation.[2] Then:

Definition 1: The rate of innovation ρ is the average number of degrees of freedom per unit of time, or, with $C_x(t_0, t_1)$ giving the number of degrees of freedom of $x(t)$ over the interval $[t_0, t_1]$,

$$\rho = \lim_{\tau\to\infty} \frac{1}{\tau} C_x(-\frac{\tau}{2}, \frac{\tau}{2}). \tag{4}$$

Definition 2: A signal with finite rate of innovation is such that $\rho < \infty$.

If we consider finite length or periodic signals of length τ, then the number of degrees of freedom is finite, and the rate of innovation is $1/\tau\, C_x[0, \tau]$.

Bandlimited signals with frequency support $[-\pi/T, \pi/T]$ have a rate of innovation $\rho = 1/T$ since they are uniquely specified by samples taken every T seconds. Bandpass signals having support $(-\omega_0 - \Delta, -\omega_0] \cup [\omega_0, \omega_0 + \Delta]$ have a rate of innovation $\rho = \Delta/\pi$, since appropriate demodulation allows sampling with a period $T = 2\pi/2\Delta$.

If we consider discrete-time sequences, then general sequences have a (normalized) rate of innovation of 1 (one degree of freedom per sample). If the underlying sampling rate is taken into account, then we have again a rate of innovation $\rho = 1/T$.

A stream of Diracs in discrete-time with K locations over an interval of size N has a rate

[2] In the sequel, we consider real signals unless specified otherwise.

of innovation of the order K/N,[3] but these are degrees of freedom over the integers. A piecewise polynomial in discrete-time has thus a combination of integer and real degrees of freedom.

One can also define a **local** rate of innovation with respect to a moving window of size τ.

Definition 3: Given a window of size τ, the local rate of innovation at time t is

$$\rho_\tau(t) = \frac{1}{\tau} C_x(t - \tau/2, t + \tau/2).$$

In this case, one is often interested in the maximal local rate, or $\rho_m(\tau)$

$$\rho_m(\tau) = \max_{t \in \mathbf{R}} \rho_\tau(t).$$

As $\tau \to \infty$, $\rho_m(\tau)$ tends to ρ. To illustrate the differences between ρ and ρ_m, consider again the Poisson process with expected interarrival time $1/\mu$. The rate of innovation ρ is given by μ. However, for any finite τ, there is no bound on $\rho_m(\tau)$, even though its expected value is μ.

While one can define many parametric signals which have a finite rate of innovation, in the sequel we will concentrate on streams of Diracs and piecewise polynomials which are classes for which we are able to give sampling theorems and reconstruction formulae.

Combinations of bandlimited signals and piecewise polynomials are also of interest, as are filtered versions of stream of Diracs.

III. Discrete-time periodic signals with finite rate of innovation

The discrete-time periodic signals we consider are streams of weighted Diracs and piecewise polynomials. Through appropriate differentiation, piecewise polynomials can be reduced to streams of Diracs, so we begin with these.

[3] Actually, slightly less because there can only be one Dirac at any one location.

A. Stream of Diracs

Consider a discrete-time periodic signal, with one period given by

$$\mathbf{x} = (x[0], x[1], \ldots, x[N-1])^T \tag{5}$$

and containing K weighted Diracs at locations $\{n_0, n_1, \ldots, n_{K-1}\}$, $n_k \in [0, N-1]$ and $K < \lfloor N/2 \rfloor$,

$$x[n] = \sum_{k=0}^{K-1} c_k \, \delta[n - n_k], \tag{6}$$

where $\delta[n]$ is the Kronecker delta and equal to 1 if $n = 0$ and 0 if $n \neq 0$.

Denote by $\mathbf{X} = (X[0], X[1], \ldots, X[N-1])^T$ the discrete-time Fourier series (DTFS) coefficients of $\mathbf{x}$ where

$$X[m] = \sum_{k=0}^{K-1} c_k W_N^{n_k m}, \quad m = 0, \ldots, N-1 \tag{7}$$

and $W_N = e^{-i2\pi/N}$.

Consider filtering the signal $x[n]$ with a lowpass filter $\tilde{\varphi}[n] = \varphi[-n]$ with bandwidth $[-K, K]$ then the sample values $y_s[l]$ are simply a subsampled version (by M) of the filtered signal $y[n] = x[n] * \tilde{\varphi}[n]$. The DTFS coefficients of $y[n]$ are given by

$$Y[m] = \begin{cases} X[m] & \text{if } m \in [-K, K] \\ 0 & \text{else} \end{cases} \tag{8}$$

and those of the subsampled signal $y_s[l] = y[lM]$ are given by the usual subsampling formula

$$Y_s[m] = \frac{1}{M} \sum_{l=0}^{M-1} Y[(m + lN)/M]. \tag{9}$$

With appropriate re-indexing it follows that

$$Y_s[m] = \frac{1}{M} X[m], \quad m \in [-K, K]. \tag{10}$$

Figure 2 illustrates that we can recover $2K$ spectral values $X[m]$ of the original signal from the subsampled spectra of the lowpass approximation $Y_s[m]$ as long as there is no overlapping in the spectra of the lowpass approximation $Y[m]$ and this occurs only if $N/M \geq 2K$.

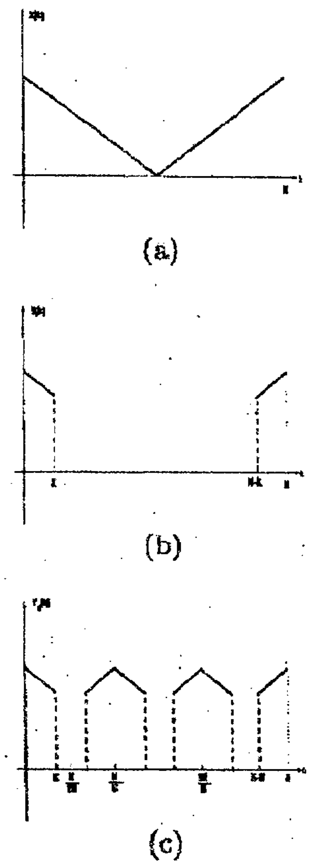


Fig. 2. (a)DTFS of stream of Diracs, $X[k], k \in [0, N]$; (b) DTFS of lowpass approximation $Y[k] = X[k], k \in [-K, K]$, 0 otherwise; (c) DTFS of lowpass approximation subsampled by $M = 3$, $Y_s[k] = 1/MX[k], k \in [-K, K]$.

This leads us to

Proposition 1: Consider a discrete-time periodic signal $x[n]$ of period N containing K weighted Diracs. Let M be an integer divisor of N satisfying $N/M \geq 2K + 1$. Consider

the discrete-time periodized sinc sampling kernel $\varphi[n] = \frac{1}{N}\sum_{m=-K}^{K} W_N^{-mn}$, that is, the inverse DTFS of the $Rect_{[-K,K]}$. Then the $N/M \in \mathbb{N}$ samples defined by

$$y_s[l] \;=\; < x[n], \varphi[n - lM] >_{circ}, \quad l = 0, \ldots, N/M - 1 \qquad (11)$$

are a sufficient representation of the signal.

Proof: We start by showing that the DTFS coefficients $X[m], m \in [-K, K]$ are sufficient to determine the stream of K weighted Diracs . Then we show that the N/M samples $y_s[l]$ are a sufficient representation of $X[m], m \in [-K, K]$.

1. Since $X[m]$ is a linear combination of K complex exponentials, u_k^m, with $u_k = W_N^{n_k}$, the locations n_k of the Diracs can be found using the annihilating filter method described in Appendix A. It suffices to determine the annihilating filter $H(z)$ whose coefficients are $(1, H[1], \ldots, H[K])$ or,

$$H(z) = 1 + H[1]\,z^{-1} + H[2]\,z^{-2} + \cdots + H[K]\,z^{-K} \qquad (12)$$

which factors as

$$H(z) = \prod_{k=0}^{K-1}(1 - z^{-1}W_N^{n_k}) \qquad (13)$$

and satisfies

$$\sum_{k=0}^{K} H[k]\,X[m-k] = 0, \quad m = 0, \ldots, N-1 \qquad (14)$$

Since $H[0] = 1$, K equations (14) will be sufficient to determine the K unknown filter coefficients $H[k], k = 1, \ldots, K$. Let $m = 1, \ldots, K$ then the system in (14) is equivalent to

$$\sum_{k=1}^{K} H[k]\,X[m-k] = -X[m], \quad m = 1, \ldots, K. \qquad (15)$$

For example take $N = 8$, $K = 3$ and let $m = 1, 2, 3$ then in matrix/vector form the system is

$$\begin{bmatrix} X[0] & X[-1] & X[-2] \\ X[1] & X[0] & X[-1] \\ X[2] & X[1] & X[0] \end{bmatrix} \cdot \begin{pmatrix} H[1] \\ H[2] \\ H[3] \end{pmatrix} = -\begin{pmatrix} X[1] \\ X[2] \\ X[3] \end{pmatrix}. \tag{16}$$

Given that these are K sinusoids the matrix in (16) is full rank ($= K$) and thus there is a unique solution $H[1], \ldots, H[K]$. The set of locations $\{n_0, n_1, \ldots, n_{K-1}\}$ are given by the the zeros of $H(z)$.

The weights of the Diracs are obtained by solving K equations in (7), let $m = 0, \ldots, K-1$, this leads to the following Vandermonde system

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ W_N^{n_0} & W_N^{n_1} & \ldots & W_N^{n_{K-1}} \\ \vdots & \vdots & \ldots & \vdots \\ W_N^{n_0(K-1)} & W_N^{n_1(K-1)} & \ldots & W_N^{n_{K-1}(K-1)} \end{bmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[K-1] \end{pmatrix} \tag{17}$$

and has a unique solution since the $n_k \neq n_l, \forall k \neq l$.

Therefore, given $2K$ contiguous DTFS coefficients

$$\{X[-K+1], X[-K+2], \ldots, X[0], \ldots, X[K]\}$$

we have found a unique set of locations $\{n_k\}_{k=0}^{K-1}$ and a unique set of weights $\{c_k\}_{k=0}^{K-1}$.

2. We need to show that $2K$ spectral values $X[m], m \in [-K, K]$ can be obtained from the N/M sample values $y_s[l]$ defined in (11).

We substitute the discrete-time periodized sinc kernel in the expression of the sample

values and we obtain the following:

$$
\begin{aligned}
y_s[l] &= \langle x[n], \varphi[n-lM] \rangle_{circ} \quad l = 0, \ldots, N/M-1 && (18)\\
&= \sum_{n=0}^{N-1} x[n]\, \varphi[n-lM] && (19)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K}^{K} W_N^{-m(n-lM)} && (20)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K}^{K} W_N^{-mn}\, W_N^{mlM} && (21)\\
&= \frac{1}{N} \sum_{m=-K}^{K} W_{N/M}^{ml} \underbrace{\sum_{n=0}^{N-1} x[n]\, W_N^{-nm}}_{X[-m]} && (22)\\
&= \frac{1}{N} \sum_{m=-K}^{K} X[-m]\, W_{N/M}^{ml}. && (23)
\end{aligned}
$$

If we calculate the DTFS coefficients of the sample values $y_s[l]$ we obtain an expression in terms of the DTFS of the signal,

$$
\begin{aligned}
Y_s[k] &= \sum_{l=0}^{N/M-1} y_s[l]\, W_{N/M}^{lk}, \quad k = 0, \ldots, N/M-1 && (24)\\
&= \frac{1}{N} \sum_{l=0}^{N/M-1} \sum_{m=-K}^{K} X[-m]\, W_{N/M}^{ml} W_{N/M}^{lk} && (25)\\
&= \frac{1}{N} \sum_{m=-K}^{K} X[-m] \underbrace{\sum_{l=0}^{N/M-1} W_{N/M}^{l(k+m)}}_{= \begin{cases} N/M & \text{if } k+m=0 \\ 0 & \text{otherwise} \end{cases}} && (26)\\
&= \frac{1}{M} X[k], \quad k = 0, \ldots, \min\{K, N/M-1\} && (27)\\
\Rightarrow X[k] &= M Y_s[k], \quad k = 0, \ldots, K && (28)
\end{aligned}
$$

by hypothesis, $N/M \geq 2K+1 > K$. Since we are dealing with real signals the DTFS is Hermitian, that is, $X[-k] = X^*[k]$, $k = 0, \ldots, K$, so we have the $2K+1$ spectral

values $X[k]$, $k \in [-K, K]$ obtained from the N/M DTFS coefficients of the sample values $y_s[l]$. Therefore we have a sufficient number of spectral values which uniquely define the stream of weighted Diracs.

■

Figure 3 illustrates in time and frequency domain the sampling of a discrete-time periodic stream of Diracs with period $N = 256$ and $K = 15$ weighted Diracs. The signal is perfectly reconstructed within machine precision, $MSE = 10^{-11}$.

Note that in the proof of Proposition 1 the locations of the Diracs are determined by finding the roots of the annihilating filter $H(z)$. If the locations are bunched up or there are a large number of Diracs then finding the roots of the polynomial is numerically unstable. An alternative method that is commonly used in error correction coding involves extrapolating the $N-K$ spectral values of the signal using K first spectral $X[k]$, $k = 1, \ldots, K$ components and the error locating polynomial which in our case corresponds to the annihilating filter $H[k], k = 1, \ldots, K$,

$$X[k] = -\sum_{l=1}^{K} H[l]\, X[k-l], \quad k = K+1, \ldots, N-K. \tag{29}$$

Consider a signal of length $N = 64$ where there are $K = 16$ Diracs in an interval of size $2K$, see Figure 4. Figure 5 compares the relative reconstruction error between the root finding method and the spectral extrapolation method for different values of K.

B. *Piecewise polynomials of degree R*

The previous result on the stream of Diracs is extended to piecewise polynomials. Consider a discrete-time periodic piecewise polynomial defined by [4]

$$x[n] = \frac{1}{R!} \sum_{k=0} c_k \, (n - n_k)_+^R \tag{30}$$

[4] $n_+ = n$, if $n \geq 0$, and 0 else.

of period N with K pieces each with maximum degree R. Suppose a discrete-time difference operator $d[n] = \delta[n] - \delta[n-1]$ is applied $R+1$ times to the piecewise polynomial signal. The differentiated signal $x^{R+1}[n]$ in frequency domain is

$$X^{(R+1)}[m] = (D[m])^{R+1}\, X[m], \quad m = 0, \ldots, N-1 \tag{31}$$

where $D[m] = 1 - W_N^m$ is the DTFS of the discrete-time difference operator. This results in putting to zero all the polynomial pieces. Assume there are discontinuities between pieces (but no Diracs), then K transitions can lead to at most $K(R+1)$ weighted Diracs and thus the rate of innovation is $\rho = 2K(R+1)/N$. From Proposition 1 we can uniquely recover the $K(R+1)$ Diracs from $2K(R+1)$ DTFS coefficients of the differentiated signal $X^{(R+1)}[k]$. The piecewise polynomial signal is reconstructed by applying the inverse discrete-time difference operator $R+1$ times on the stream of weighted Diracs. The discrete-time difference operator $d[n]$ is a singular operator (since $D[0] = 0$) and so we define the inverse discrete-time difference operator as $D^{-1}[m] = 0$ for $m = 0$ and $D^{-1}[m] = (1 - W_N^m)^{-1}$ for $m = 1, \ldots, N-1$. Hence instead of using the sinc sampling kernel $\varphi[n]$ we will use the derivative sinc sampling kernel defined by $\psi[n] = (\underbrace{d * d * \cdots * d}_{R+1} * \varphi)[n]$ which has at least $R+1$ zeros at the origin $z = 1$. Then the DTFS of $\psi[n]$ is

$$\Psi[m] \;=\; (1 - W_N^m)^{R+1}\, \Phi[m], \quad m = 0, \ldots, N-1 \tag{32}$$

where $\Phi[m]$ is the $Rect_{[-K(R+1),K(R+1)]}$ function. This brings us to the following theorem.

Theorem 1: Consider a discrete-time periodic piecewise polynomial signal of period N with K pieces of degree R and with zero mean. [5] Let M be an integer and a divisor of N such that $N/M \geq (2K(R+1)+1)$. Take a sampling kernel $\psi[n]$ with DTFS coefficients defined in (32). Then we can recover the signal from the $N/M \in \mathbb{N}$ samples

$$y_s[l] \;=\; < x[n], \psi[n - lM] > \quad l = 0, \ldots, N/M - 1. \tag{33}$$

[5] We consider zero mean signals since $D \circ D^{-1}$ is a projector on the space of signals having zero mean.

Proof: First we show that $2K(R+1)$ DTFS coefficients of the signal, $X[m], m \in [-K(R+1), K(R+1)]$ are sufficient to determine the piecewise polynomial signal, $x[n]$. Then we show that the N/M samples $y_s[l]$ are sufficient to determine the $(2K(R+1)+1)$ values $X[m]$.

1. If we have the DTFS coefficients $X[m], m \in [-K(R+1), K(R+1)]$ then from (31) we have the DTFS coefficients of the $(R+1)$th discrete-time differentiated signal, $X^{(R+1)}[m]$. From Proposition 1 these are sufficient to reconstruct the stream of $K(R+1)$ Diracs. Thus, the signal is recovered by applying $R+1$ times the inverse discrete-time difference operator, $d^{-1}[n]$, on the stream of Diracs, that is,

$$x[n] = \big(\underbrace{d^{-1} * d^{-1} * \cdots * d^{-1}}_{R+1} * x^{(R+1)}\big)[n].$$

2. Similar to the second part in the proof of Proposition 1 we expand the inner product between the piecewise polynomial signal and the differentiated sinc sampling kernel:

$$\begin{aligned}
y_s[l] &= < x[n], \psi[n-lM] >, \quad l = 0, \ldots, N/M-1 && (34)\\
&= \sum_{n=0}^{N-1} x[n]\, \psi[n-lM] && (35)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, W_N^{-m(n-lM)} && (36)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, W_N^{-mn}\, W_N^{mlM} && (37)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1} W_{N/M}^{ml} \underbrace{\sum_{n=0}^{N-1} x[n]\, W_N^{-nm}}_{X[-m]} && (38)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, X[-m]\, W_{N/M}^{ml}. && (39)
\end{aligned}$$

Taking the DTFS of the sample values $y_s[l]$ we obtain

$$
\begin{aligned}
Y_s[k] &= \sum_{l=0}^{N/M-1} y_s[l]\, W_{N/M}^{lk}, \quad k=0,\ldots,N/M-1 &(40)\\
&= \frac{1}{N}\sum_{l=0}^{N/M-1}\sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, X[-m]\, W_{N/M}^{ml} W_{N/M}^{lk} &(41)\\
&= \frac{1}{N}\sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, X[-m] \underbrace{\sum_{l=0}^{N/M-1} W_{N/M}^{l(k+m)}}_{=\begin{cases} N/M & \text{if } k+m=0\\ 0 & \text{otherwise}\end{cases}} &(42)\\
&= \frac{1}{M}(1-W_N^{-k})^{R+1}\, X[k], \quad k=0,\ldots,\min\{N/M, K(R+1)\} &(43)
\end{aligned}
$$

$$
\Rightarrow X[k] = \begin{cases} M\left[(1-W_N^{-k})^{R+1}\right]^{-1} Y_s[k] & \text{for } k=1,\ldots,K(R+1)\\ 0 & \text{for } k=0 \end{cases}. \qquad (44)
$$

Since $N/M \geq (2K(R+1)+1)$ we have a sufficient representation for the spectral values of the signal. This completes the proof.

■

Figure 6 illustrates the reconstruction of a discrete-time periodic piecewise linear signal of period $N = 1024$ with $K = 6$ pieces.

IV. Continuous-time periodic signals with finite rate of innovation

We derive now the equivalent results but for continuous-time periodic signals, again building up from a stream of Diracs to piecewise polynomials. We will put in evidence the common points.

A. Stream of Diracs

Consider a continuous-time periodic signal $x(t)$ of period τ containing K weighted Diracs at locations $\{t_k\}_{k=0}^{K-1}$ with $t_k \in [0, \tau)$, or

$$\begin{aligned} x(t) &= \sum_{n \in \mathbb{N}} c_n \, \delta(t - t_n) \\ &= \sum_{n \in \mathbb{N}} \sum_{k=0}^{K-1} c_k \, \delta(t - (t_k + n\tau)) \end{aligned} \tag{45}$$

since $t_{n+K} = t_n + \tau$ and $c_{n+K} = c_n$ for all $n \in \mathbb{N}$.

The continuous-time Fourier series (CTFS) coefficients of $x(t)$ are defined by

$$\begin{aligned} X[m] &= \frac{1}{\tau} \int_0^\tau x(t) \, e^{-i2\pi tm/\tau} \, dt, \quad m \in \mathbb{Z} \\ &= \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \, e^{-i2\pi t_k m/\tau}. \end{aligned} \tag{46}$$

If the signal $x(t)$ is convolved with a sinc filter of bandwidth $[-K, K]$ then we have a lowpass approximation $y(t)$ given by

$$y(t) = \sum_{m=-K}^{K} X[m] \, e^{i2\pi mt/\tau}. \tag{47}$$

Suppose the lowpass approximation $y(t)$ is sampled at multiples of T, we obtain $\tau/T \in \mathbb{N}$ samples defined by

$$y_s[l] = y(lT) = \sum_{m=-K}^{K} X[m] \, e^{i2\pi mlT/\tau}, \quad l = 0, \ldots, \tau/T - 1. \tag{48}$$

Similar to the discrete-time case as long as the number of samples is larger than the number of values in the spectral support of the lowpass signal, that is, $\frac{\tau}{T} \geq 2K + 1$, (48) can be used to recover $2K + 1$ values of $X[m]$. Thus we can state:

Proposition 2: Consider a continuous-time periodic stream of K weighted Diracs with period τ and a continuous-time periodic sinc sampling kernel $\varphi(t)$ with bandwidth $[-K, K]$.

Taking a sampling period T such that $\tau/T \in \mathbb{N}$ and $\tau/T \geq 2K+1$. Then the samples defined by

$$y_s[l] = < x(t), \varphi(t - lT) >, \quad l = 0, \ldots, \tau/T - 1 \tag{49}$$

are a sufficient representation of $x(t)$.

Proof: Similar to the discrete-time case first we show that $2K+1$ CTFS coefficients $X[m]$ are sufficient to find the locations and the weights of the Diracs. From (46) we have that the CTFS coefficients $X[m]$ are linear combinations of complex exponentials. Thus to find the locations t_k we need to find the annihilating filter $\mathbf{H} = (1, H[1], H[2], \ldots, H[K])$ such that

$$\mathbf{H} *_c \mathbf{X} = 0. \tag{50}$$

This is the same Toeplitz system as in (14) considered in Sec. III-A and therefore a solution exists. Factoring the $z-$transform of H, or $H(z) = \sum_{k=0}^{K} H[k]\, z^{-k}$, into

$$H(z) = \prod_{k=0}^{K-1} (1 - z^{-1} u_k), \tag{51}$$

we then find the K locations $\{t_0, t_1, \ldots, t_{K-1}\}$ from the zeros of $H(z)$, that is, from

$$u_k = e^{-i2\pi t_k/\tau}. \tag{52}$$

Given the locations $\{t_k\}_{k=0}^{K-1}$ and K values $X[m], m = 0, \ldots, K-1$, we find the weights $\{c_k\}_{k=0}^{K-1}$ of the Diracs by solving the Vandermonde system in (46). Since the locations t_k are distinct, $t_k \neq t_l, \forall k \neq l$, the Vandermonde system admits a solution.

The second part of the proof consists in showing that the τ/T samples $y_s[l]$ are sufficient to determine the CTFS coefficients $X[m], m \in [-K, K]$. We substitute the continuous-time periodic sinc function $\varphi(t)$ with bandwidth $[-K, K]$ defined by

$$\varphi(t) = \sum_{m=-K}^{K} e^{i2\pi m t}. \tag{53}$$

in (49) and we obtain

$$\begin{aligned}
y_s[l] &= \langle x(t), \varphi(t-lT) \rangle, \quad l = 0,\ldots,\tau/T-1 && (54)\\
&= \int_0^{\tau} x(t) \sum_{m=-K}^{K} e^{i2\pi m(t-lT)/\tau}\, dt && (55)\\
&= \sum_{m=-K}^{K} e^{-i2\pi ml/(\tau/T)} \underbrace{\int_0^{\tau} x(t) e^{i2\pi mt/\tau}\, dt}_{\tau X[-m]} && (56)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, e^{-i2\pi mlT/\tau}. && (57)
\end{aligned}$$

Note that $y_s[l]$ is periodic with period τ/T, thus the DTFS coefficients are $Y_s[k] = TX[k], k = 0,\ldots,\tau/T-1$. Since $\tau/T \geq 2K+1$, we have a sufficient number of samples that determine the CTFS $X[m], m \in [-K, K]$. ■

B. Piecewise polynomials of degree R

Here we consider continuous-time periodic piecewise polynomial signal of period τ, containing K pieces of maximum degree R and $R-1$ continuous derivatives, $\mathcal{C}^{R-1}$,

$$x(t) = \frac{1}{R!} \sum_{k=0}^{K-1} c_k \,(t-t_k)_+^R, \quad t \in [0,\tau]. \tag{58}$$

We differentiate the signal $R+1$ times and we obtain a continuous-time stream of K weighted Diracs, $x^{(R+1)}(t)$. The CTFS of the derivative operator is defined by $D[m] = i2\pi m, m \in \mathbb{Z}$ and therefore the CTFS coefficients of the differentiated signal $x^{(R+1)}(t)$ are equal to

$$X^{(R+1)}[m] = (i2\pi m)^{R+1} X[m], \quad m \in \mathbb{Z}. \tag{59}$$

From Proposition 2 we can recover the continuous-time periodic stream of K Diracs from the CTFS coefficients, $X^{(R+1)}[m], m \in [-K, K]$. Therefore we can sample the signal with the differentiated sinc sampling kernel whose CTFS coefficients are defined by

$$\Psi[m] = (i2\pi m)^{R+1}\, \Phi[m], \quad m \in \mathbb{Z} \tag{60}$$

where $\Phi[m] = Rect_{[-K,K]}$ is the CTFS of the continuous-time periodized sinc sampling kernel.

Theorem 2: Consider a continuous-time periodic piecewise polynomial signal $x(t)$ with period τ, containing K pieces of maximum degree R, belonging to $\mathcal{C}^{R-1}$ and having zero mean. Consider a sampling kernel $\psi(t)$ with its CTFS coefficients defined in (60). Let $\tau/T \in \mathbb{N}$ and $\tau/T \geq 2K+1$. Then $x(t)$ can be uniquely recovered from the τ/T samples

$$y_s[l] = < x(t), \psi(t - lT) >, \quad l = 0, \ldots, \tau/T - 1. \tag{61}$$

Proof: Similar to the proof of Theorem. 1, we first show that CTFS coefficients $X[m], m \in [-K, K]$ are sufficient to determine the piecewise polynomial signal, $x(t)$. Then we show that the τ/T samples $y_s[l]$ are sufficient to determine the values $X[m], m \in [-K, K]$.

1. If we have the CTFS coefficients $X[m], m \in [-K, K]$ then from (59) we have the CTFS coefficients of the $(R+1)$th differentiated signal, $X^{(R+1)}[m]$. From Proposition 2 these are sufficient to reconstruct the stream of K Diracs. Thus, the signal is recovered by integrating $R+1$ times the stream of Diracs, that is,

$$x(t) = \underbrace{\int\int \cdots \int}_{R+1} x^{(R+1)}(t)\, dt\, dt \cdots dt$$

or in frequency domain from (59)

$$X[m] = (D^{-1}[m])^{R+1} X^{(R+1)}[m], \quad m \in \mathbb{Z}/\{0\} \tag{62}$$

$$= (i2\pi m)^{-(R+1)} X^{(R+1)}[m], \quad m \in \mathbb{Z}/\{0\} \tag{63}$$

with $D^{-1}[m] = 0$ for $m = 0$ and thus

$$x(t) = \sum_{m \in \mathbb{Z}} X[m]\, e^{i2\pi mt/\tau}.$$

2. Similar to the second part in the proof of Proposition 2 we expand the inner product between the piecewise polynomial signal and the differentiated sinc sampling kernel

defined by $\psi(t) = \sum_{m=-K}^{K} (i2\pi m)^{R+1} e^{i2\pi m t/\tau}$. That is, the sample values are

$$\begin{aligned}
y_s[l] &= < x(t), \psi(t - lT) >, \quad l = 0, \dots, \tau/T - 1 && (64)\\
&= \int_0^{\tau} x(t) \sum_{m=-K}^{K} (i2\pi m)^{R+1} e^{i2\pi m(t-lT)/\tau}\, dt && (65)\\
&= \sum_{m=-K}^{K} (i2\pi m)^{R+1}\, e^{-i2\pi m lT/\tau} \underbrace{\int_0^{\tau} x(t)\, e^{i2\pi m t/\tau}\, dt}_{\tau X[-m]} && (66)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1}\, e^{-i2\pi m l/(\tau/T)}. && (67)
\end{aligned}$$

Since $y_s[l]$ is periodic with period τ/T, the DTFS coefficients of $y_s[l]$ are given by

$$\begin{aligned}
Y_s[k] &= \sum_{l=0}^{\tau/T-1} y_s[l]\, e^{-i2\pi k l/(\tau/T)} && (68)\\
&= \tau \sum_{l=0}^{\tau/T-1} \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1}\, e^{-i2\pi m l/(\tau/T)}\, e^{-i2\pi k l/(\tau/T)} && (69)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1} \underbrace{\sum_{l=0}^{\tau/T-1} e^{-i2\pi(k+m)l/(\tau/T)}}_{=\begin{cases} \tau/T & \text{if } k+m=0 \\ 0 & \text{otherwise} \end{cases}} && (70)\\
&= \frac{\tau^2}{T} (-i2\pi m)^{R+1}\, X[k] && (71)
\end{aligned}$$

Therefore the CTFS coefficients of the signal are obtained by the DTFS coefficients of the samples values $Y_s[m], m = 0, \dots, \tau/T - 1$ and are defined by

$$X[m] = \begin{cases} T\, Y_s[m]/(\tau^2 (-i2\pi m)^{R+1}) & \text{for } m = 1, \dots, \tau/T - 1 \\ 0 & \text{for } m = 0 \end{cases} \qquad (72)$$

Since $\tau/T \geq 2K+1$ the sample values are a sufficient representation of the spectral values of the signal. This completes the proof.

■

Note that removing the restriction $x(t) \in C^{K-1}$ leads to the same result as in Theorem 1.

V. Finite length signals with finite rate of innovation

A finite length signal with finite rate of innovation ρ clearly has a finite number of degrees of freedom. The question of interest is: Given a sampling kernel with *infinite support*, is there a *finite set of samples* that uniquely specifies the signal? In the following sections we will sample signals with finite number of weighted Diracs with infinite support sampling kernels such as the sinc and Gaussian.

A. Sinc sampling kernel

Consider a continuous-time signal with a finite number of weighted Diracs

$$x(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k) \tag{73}$$

and an infinite length sinc sampling kernel, see Figure 7. The sample values are obtained by filtering the signal with a $\text{sinc}(t/T), t \in \mathbb{R}$, sampling kernel. This is equivalent to taking the inner product between the signal and a shifted version of the sinc

$$y_n = < x(t), \text{sinc}(t/T - n) > \tag{74}$$

where $\text{sinc}(t) = \sin(\pi t)/\pi t$. The question that arises is: How many of these samples do we need to recover the signal? The signal has $2K$ degrees of freedom, K from the weights and K from the locations of the Diracs and thus N samples, $N \geq 2K$, will be sufficient to recover the signal. Similar to the previous cases, the reconstruction method will require solving two systems of linear equations: one for the locations of the Diracs and the second for the weights of the Diracs. These systems admit solutions if the following conditions are satisfied:

[C1] Rank(V) $< K+1$ where $v_{nk} = \Delta^K\left((-1)^n n^k y_n\right)$ and $\mathbf{V} \in \mathbb{R}^{(N-K)\times(K+1)}$;

[C2] Rank(A) $= K$ where $a_{nk} = \frac{\sin(\pi t_k/T)}{\pi(t_k/T-n)}$ and $\mathbf{A} \in \mathbb{R}^{K\times K}$.

Theorem 3: Given a finite stream of K weighted Diracs and a sinc sampling kernel $\text{sinc}(t/T)$. If conditions [C1] and [C2] are satisfied then N samples with $N \geq 2K$

$$y_n = < x(t), \text{sinc}(t/T-n) > \tag{75}$$

are a sufficient representation of the signal.

Proof: Taking the inner products between the signal and shifted versions of the sinc sampling kernel yields a set of N samples

$$\begin{aligned}
y_n &= < x(t), \text{sinc}(t/T-n) >, \quad n = 0, \dots, N-1 && (76)\\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{K-1} c_k\, \delta(t-t_k)\, sinc(t/T-n)\, dt && (77)\\
&= \sum_{k=0}^{K-1} c_k \,\text{sinc}\left(t_k/T-n\right) && (78)\\
&= \sum_{k=0}^{K-1} \frac{c_k \sin\left(\pi t_k/T - \pi n\right)}{\pi\left(t_k/T-n\right)} && (79)\\
&= (-1)^n \sum_{k=0}^{K-1} \frac{c_k \sin\left(\pi t_k/T\right)}{\pi\left(t_k/T-n\right)} && (80)\\
\Leftrightarrow (-1)^n y_n &= \frac{1}{\pi}\sum_{k=0}^{K-1} c_k \sin\left(\pi t_k/T\right) \cdot \frac{1}{\left(t_k/T-n\right)} && (81)
\end{aligned}$$

The denominator of the previous expression (81) can be rewritten as follows:

$$\frac{1}{\left(t_k/T-n\right)} = \frac{\prod_{l\neq k}\left(t_l/T-n\right)}{\prod_{l=0}^{K-1}\left(t_l/T-n\right)} = \frac{P_k(n)}{P(n)} \tag{82}$$

where $P(u)$ is a polynomial of degree K with zeros at all values of t_k/T,

$$P(u) = \prod_{l=0}^{K-1}\left(t_l/T-u\right) = \sum_{k=0}^{K} p_k\, u^k \tag{83}$$

and the $P_k(u)$ is a polynomial of degree $K-1$ and has zeros at all locations except at location t_k

$$P_k(u) \;=\; \prod_{l\neq k}(t_l/T-u). \tag{84}$$

Therefore if the coefficients of the polynomial $P(u)$ are determined then the locations of the Diracs are simply the K roots of $P(u)$. We can now find an equivalent expression to (81) in terms of the interpolating polynomials:

$$(-1)^n\,P(n)\,y_n \;=\; \frac{1}{\pi}\sum_{k=0}^{K-1} c_k \sin\left(\pi t_k/T\right) P_k(n). \tag{85}$$

Note that the right-hand side of (85) is a polynomial of degree $K-1$ in the variable n, applying K finite differences makes the left-hand side vanish,[6] that is,

$$\begin{aligned} & \Delta^K\big((-1)^n\,P(n)\,y_n\big) && = 0, \quad n = K,\ldots,N-1 && (86)\\ \Leftrightarrow\quad & \textstyle\sum_{k=0}^{K} p_k \underbrace{\Delta^K\left((-1)^n\,n^k\,y_n\right)}_{v_{nk}} && = 0 && (87)\\ \Leftrightarrow\quad & \mathbf{V}\cdot\mathbf{p} && = 0 && (88)\end{aligned}$$

where the matrix $\mathbf{V}$ is an $(N-K)\times(K+1)$ matrix and admits a solution when $N-K\geq K$ and the rank($\mathbf{V}$) is less than $K+1$, that is, condition [C1]. Therefore (87) can be used to find, up to a normalization, the $K+1$ unknowns p_k which lead to the K locations t_k. Once the K locations t_k are determined the weights of the Diracs c_k are found by solving the system in (81) for $n=0,\ldots,K-1$. Since $t_k\neq t_l, \forall k\neq l$, the system admits a solution from condition [C2]. ■

Note that the result does not depend on T. In practice if T is not chosen appropriately then the matrices $\mathbf{V}$ may be ill-conditioned. Figure 8(a) illustrates the conditioning of the matrix $\mathbf{V}$ is the least for T close to 0.5 and that the matrix $\mathbf{A}$ is well-conditioned on average.

By choosing more adequately the interpolating polynomials, for example by taking the Lagrange polynomials, we may reduce the conditioning of the matrix $\mathbf{V}$, but this remains to be investigated. The algorithm is as follows:

Algorithm 1: Finite length stream of Diracs sampled with a sinc sampling kernel

Given $y_n =< x(t), \mathrm{sinc}(t/T - n) >, \quad n = 0, 1, \dots, N-1;$

Calculate $v_{nk} = \Delta^K\big((-1)^n n^k y_n\big), \quad n = K, \dots, N-1, \quad k = 0, \dots, K;$

Solve the linear system $\mathbf{V} \cdot \mathbf{p} = 0 \longrightarrow \{p_0, p_1, \dots, p_K\};$

Find the K roots of $P(u) = \sum_{k=0}^{K} p_k u^k \longrightarrow \{t_0/T, t_1/T, \dots, t_{K-1}/T\};$

Calculate $a_{nk} = \frac{\sin\left(\pi t_k/T\right)}{\pi\left(t_k/T - n\right)}, \quad n = 0, 1, \dots, N-1\ ;$

Calculate $Y_n = (-1)^n P(n)\, y_n, \quad n = 0, 1, \dots, N-1;$

Solve the linear system $\mathbf{A} \cdot \mathbf{c} = \mathbf{Y} \longrightarrow \{c_0, c_1, \dots, c_{K-1}\}.$

This method can be extended to piecewise polynomials, similarly to Theorem. 2. Also, there is an obvious equivalent for discrete-time signals in $\ell^2(\mathbb{Z})$ and discrete-time sinc kernels.

B. Gaussian sampling kernel

Consider sampling the same signal as in (73) but this time with a Gaussian sampling kernel, $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}$, see Figure 9. Similar to the sinc sampling kernel, the samples are obtained by filtering the signal with a Gaussian kernel. Since there are $2K$ unknown variables we show next that N samples with $N \geq 2K$ are sufficient to represent the signal.

Theorem 4: Given a finite stream of K weighted Diracs and a Gaussian sampling kernel $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}$. If $N \geq 2K$ then the N sample values

$$y_n =< x(t), \varphi_\sigma(t/T - n) > \tag{89}$$

[6] Note that the K finite difference operator plays the same role as the annihilating filter in the previous chapter.

are sufficient to reconstruct the signal.

Proof: The sample values are given by

$$
\begin{aligned}
y_n &= \langle x(t), e^{-(t/T-n)^2/2\sigma^2} \rangle, \quad n = 0, \ldots, N-1 && (90)\\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{K-1} c_k\, \delta(t-t_k)\, e^{-(t/T-n)^2/2\sigma^2}\, dt && (91)\\
&= \sum_{k=0}^{K-1} c_k\, e^{-(t_k/T-n)^2/2\sigma^2}. && (92)
\end{aligned}
$$

We expand (92) and regroup the terms so as to have variables that depend solely on n and solely on k. We obtain

$$
y_n = \sum_{k=0}^{K-1} \left(c_k\, e^{-t_k^2/2\sigma^2 T^2}\right) \cdot e^{n t_k/\sigma^2 T} \cdot e^{-n^2/2\sigma^2} \qquad (93)
$$

which is equivalent to

$$
Y_n = \sum_{k=0}^{K-1} a_k\, u_k^n \qquad (94)
$$

where we let $Y_n = e^{n^2/2\sigma^2}\, y_n$, $a_k = c_k\, e^{-t_k^2/2\sigma^2 T^2}$ and $u_k = e^{t_k/\sigma^2 T}$. Note that we reduced the expression Y_n to a linear combination of real exponentials. This hints that the annihilating filter method described in the Section III-A seems appropriate to find the K values u_k. Let $H(z) = h_0 + h_1\, z^{-1} + \cdots + h_K\, z^{-K}$ be an annihilating filter, that is, $\mathbf{h}$ is such that

$$
\begin{aligned}
\mathbf{h} * \mathbf{Y} &= 0 && (95)\\
\Leftrightarrow \sum_{k=0}^{K} h_k\, Y_{n-k} &= 0, \quad n = K, \ldots, N-1. && (96)
\end{aligned}
$$

Note that this is a Toeplitz system with real exponential components $Y_n = e^{n^2/2\sigma^2}\, y_n$ and therefore a solution exists when the number of equations is greater than the number of unknowns, that is, $N - K \geq K$ and the rank of the system is less than $K+1$ which is the case by hypothesis. Furthermore σ must be carefully chosen otherwise the system is

ill-conditioned. If we factor $H(z) = \prod_{k=0}^{K-1}(1 - z^{-1}u_k)$ then we obtain the locations of the Diracs t_k from the roots of the polynomial $H(z)$, that is,

$$t_k = \sigma^2 T \ln u_k. \tag{97}$$

Once the values of the Diracs t_k are obtained then we solve for a_k the Vandermonde system in (94) for which a solution exists since $u_k \neq u_l, \forall k \neq l$. The weights of the Diracs are simply given by

$$c_k = a_k\, e^{t_k^2/2\sigma^2 T^2}. \tag{98}$$

■

Here unlike in the sinc case, we have an almost local reconstruction because of the exponential decay of the Gaussian sampling kernel which brings us to the next topic.

VI. Infinite length signals with finite local rate of innovation

In this section we consider the dual problem of Sec. V, that is, *infinite* length *signals* $x(t), t \in \mathbb{R}^+$ with a finite *local* rate of innovation and sampling kernels with *compact support*. In particular, the $\beta-$splines of different degree d are considered [8]

$$\varphi_d(t) = (\varphi_{d-1} * \varphi_0)(t), \quad d \in \mathbb{N}^+ \tag{99}$$

where $\varphi_0(t)$ is the box spline defined by

$$\varphi_0(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{else} \end{cases}. \tag{100}$$

We develop local reconstruction algorithms which depend on moving intervals equal to the size of the support of the sampling kernel.[7] The advantage of local reconstruction algorithms is that their complexity does not depend on the length of the signal. We begin by considering bilevel signals, followed by piecewise polynomial signals.

[7] The size of the support of $\varphi_d(t/T)$ is equal to $(d+1)T$.

A. Bilevel signals

Consider an infinite length continuous-time signal $x(t), t \in \mathbb{R}^+$ which takes on two values, 0 and 1, with initial condition $x(t)\big|_{t=0} = 1$ with a finite local rate of innovation, ρ. These are called bilevel signals and are completely represented by their transition values t_k. For example, binary signals such as amplitude or position modulated pulses or PAM, PPM signals [3], see Fig. 10.

Suppose a bilevel signal is sampled with a box spline $\varphi_0(t/T)$. Then the sample values are given by the inner products between the bilevel signal and the box function,

$$y_n =< x(t), \varphi_0(t/T - n) >= \int_{-\infty}^{\infty} x(t)\, \varphi_0(t/T - n)\, dt. \tag{101}$$

It can bee seen in Fig. 11 that the sample value y_n corresponds to the area occupied by the signal in the interval $[nT, (n+1)T]$. Thus if there is at most one transition per box then we can recover the transition from the sample. This leads us to

Proposition 3: A bilevel signal $x(t), t > 0$, with initial condition $x(t)\big|_{t=0} = 1$, is uniquely determined from the samples $y_n =< x(t), \varphi_0(t/T - n) >$ where $\varphi_0(t)$ is the box spline defined in (100) if and only if there is at most one transition t_k in each interval $[nT, (n+1)T]$.

Proof: For simplicity let $T = 1$. Consider an interval $[n, n+1]$ and suppose $x(n) = 1$. First we show sufficiency followed by necessity .

$\Leftarrow$: If there are 0 transitions in the interval $[n, n+1]$ then the area under the bilevel signal, or the sample value, is $y_n = 1$ since we supposed that $x(t)\big|_{t=n} = 1$. If there is one transition in $[n, n+1]$ then the sample value is equal to

$$\begin{aligned} y_n &= < x(t), \varphi_0(t - n) >= \int_{-\infty}^{\infty} x(t)\, \varphi_0(t - n)\, dt && (102)\\ &= \int_{n}^{n+1} x(t)\, dt = \int_{n}^{t_k} 1\, dt = t_k - n && (103) \end{aligned}$$

This implies that $t_k = y_n - n$. Similarly if $x(n) = 0$ then we have $t_k = n + 1 - y_n$. Therefore we can uniquely determine the signal in the interval $[n, n+1]$.

$\Rightarrow$: Necessity is shown by counterexample.

Suppose $x(n) = 1$ and there are two transitions t_k, t_{k+1} in the interval $[n, n+1]$ then the sample value is equal to

$$\begin{aligned} y_n &= \int_n^{n+1} x(t)\, dt = \int_n^{t_k} 1\, dt + \int_{t_{k+1}}^{n+1} 1\, dt && (104)\\ &= t_k - n + n + 1 - t_{k+1} = t_k - t_{k+1} + 1. && (105)\end{aligned}$$

That is, there is one equation with two unknowns and therefore insufficient samples to determine both transitions. Thus there must be at most one transition in an interval $[n, n+1]$ to uniquely define the signal.

■

Now consider shifting the bilevel signal by an unknown shift ϵ, see Fig. 12, then there will be two transitions in an interval of length T and one box function will not be sufficient to recover the transitions. Suppose we double the sampling rate, then the support of the box sampling kernel is doubled and we have two sample values y_n, y_{n+1} covering the interval $[nT, (n+1)T]$ but these values are identical (see their areas). Therefore increasing the sampling rate is still insufficient.

This brings us to consider a sampling kernel not only with a larger support but with added information. For example, the hat spline function $\varphi_1(t/T)$ defined by

$$\varphi_1(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{else} \end{cases} \qquad (106)$$

leads to sample values defined by $y_n =< x(t), \varphi_1(t/T - n) >$ or

$$y_n = \int_{(n-1)T}^{nT} x(t)\big(1 + t/T - n\big)\, dt + \int_{nT}^{(n+1)T} x(t)\big(1 - (t/T - n)\big)\, dt. \qquad (107)$$

From Fig. 13 we can see that there are two sample values covering the interval $[nT, (n+1)T]$. We will show next that in this case we can uniquely determine the signal.

Proposition 4: An infinite length bilevel signal $x(t)$, with initial condition $x(0) = 1$ is uniquely determined from the samples defined by

$$y_n = < x(t), \varphi_1(t/T - n) > \tag{108}$$

where $\varphi_1(t)$ is the hat sampling kernel if and only if there are at most two transitions $t_k \neq t_j$ in each interval $[nT, (n+2)T]$.

Proof: Again, for simplicity let $T = 1$ and suppose the signal is known for $t \leq n$ and $x(t)\big|_{t=n} = 1$.

First we show sufficiency by showing the existence and uniqueness of a solution. Then we show necessity by a counterexample.

$\Leftarrow$: Similar to the box sampling kernel the sample values will depend on the configuration of the transitions in the interval $[n, n+2]$. If there are at most 2 transitions in the interval $[n, n+2]$ then the possible configurations are

$$(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)$$

where the first and second component indicate the number of transitions in the intervals $[n, n+1]$, $[n+1, n+2]$ respectively, see Fig. 14. Furthermore since the hat sampling kernel is of degree one we obtain for each configuration a quadratic system of equations with variables t_0, t_1.

$$y_n = \int_{n-1}^{n} x(t)\big(1 + t - n\big)\, dt + \int_{n}^{n+1} x(t)\big(1 - (t-n)\big)\, dt \tag{109}$$

$$y_{n+1} = \int_{n}^{n+1} x(t)\big(1 + t - (n+1)\big)\, dt + \int_{n+1}^{n+2} x(t)\big(1 - (t-(n+1))\big)\, dt. \tag{110}$$

First we show that the quadratic system of equations admits a solution and then that it is unique.

(a) Existence.

Take $n = 0$ and so the moving interval is $[0, 2]$.

The configuration $(0, 0)$ will lead to sample values $y_0 = 1$, $y_1 = 1$.

The configuration $(0, 1)$ will lead to sample values

$$y_0 = 1/2 + \int_1^{t_0} (t-1)\, dt = \frac{1}{2} {t_0}^2 + 1 - t_0 \tag{111}$$

$$y_1 = 1/2 + \int_1^{t_0} (2-t)\, dt = -\frac{1}{2} {t_0}^2 - 1 + 2t_0 \tag{112}$$

$\Rightarrow t_0 = y_0 + y_1 = 1 + \sqrt{-1 + 2y_0} = 2 - \sqrt{2 - 2y_1}$.

The configuration $(0, 2)$ will lead to sample values

$$y_0 = \frac{1}{2} {t_0}^2 + 1 - t_0 + t_1 - \frac{1}{2} {t_1}^2 \tag{113}$$

$$y_1 = -\frac{1}{2} {t_0}^2 + 1 + 2\, t_0 + \frac{1}{2} {t_1}^2 - 2\, t_1 \tag{114}$$

$\Rightarrow t_0 = \frac{(-2-2y_1+y_0^2+2y_0y_1+y_1^2)}{2(-2+y_0+y_1)}, t_1 = \frac{-(10-6y_1-8y_0+y_1^2+y_0^2+2y_0y_1)}{2(-2+y_0+y_1)}$.

The configuration $(1, 0)$ will lead to sample values

$$y_0 = -\frac{1}{2} {t_0}^2 + t_0 \tag{115}$$

$$y_1 = \frac{1}{2} {t_0}^2 \tag{116}$$

$\Rightarrow t_0 = y_0 + y_1 = 1 - \sqrt{1 - 2y_0} = \sqrt{2y_1}$.

The configuration $(1, 1)$ will lead to sample values

$$y_0 = -\frac{1}{2} {t_0}^2 + t_0 - \frac{1}{2} {t_1}^2 + t_1 \tag{117}$$

$$y_1 = \frac{1}{2} {t_0}^2 + 2 + \frac{1}{2} {t_1}^2 - 2\, t_1 \tag{118}$$

$$\Rightarrow t_0 = \frac{y_0+y_1+\sqrt{-y_1^2-2y_0y_1+4y_1-y_0^2}}{2}, t_1 = \frac{-y_1-y_0+4+\sqrt{-y_1^2-2y_0y_1+4y_1-y_0^2}}{2}$$

The configuration $(2,0)$ will lead to sample values

$$y_0 = -\frac{1}{2}{t_0}^2+1+t_0+\frac{1}{2}{t_1}^2-t_1 \tag{119}$$

$$y_1 = \frac{1}{2}{t_0}^2+1-\frac{1}{2}{t_1}^2 \tag{120}$$

$$\Rightarrow t_0 = \frac{2-2y_1+y_1^2+2y_0y_1+y_0^2-4y_0}{2(-2+y_1+y_0)}, t_1 = -\frac{6-4y_0-6y_1+y_1^2+2y_0y_1+y_0^2}{2(-2+y_1+y_0)}.$$

(b) Uniqueness.

If $y_n = 1$ and $y_{n+1} = 1$ then this implies configuration $(0,0)$.

If $y_n = 1$ and $1/2 \leq y_{n+1} \leq 1$ then the possible configurations are $(0,1)$, $(0,2)$. By hypothesis, there are at most two transitions in the interval $[n+1, n+3]$ therefore if $y_{n+2} \leq 1/2$ then the configuration in the interval $[n, n+2]$ is $(0,1)$ otherwise if $y_{n+2} \geq 1/2$ then the configuration is $(0,2)$.

If $1/2 \leq y_n \leq 1$ and $1/2 \leq y_{n+1} \leq 1$ then this implies configuration $(2,0)$.

If $1/2 \leq y_n \leq 1$ and $0 \leq y_{n+1} \leq 1/2$ then this implies configuration $(1,0)$.

$\Rightarrow$: Necessity is shown by counterexample.

Consider a bilevel signal with three transitions in the interval $[0,2]$ but with all three in the interval $[0,1]$, see Fig. 15. Then the sample values in this case are are equal to

$$y_0 = 1/2+\int_0^{t_0}(1-t)\,dt+\int_{t_1}^{t_2}(1-t)\,dt \tag{121}$$

$$= 1/2+t_0-t_1+t_2-t_0^2/2+t_1^2/2-t_2^2/2 \tag{122}$$

$$y_1 = \int_0^{t_0}t\,dt+\int_{t_1}^{t_2}t\,dt \tag{123}$$

$$= t_0^2/2-t_1^2/2+t_2^2/2. \tag{124}$$

There is no unique solution for this quadratic system of equations. Therefore there

must be at most 2 transitions in an interval $[0, 2]$.

■

Once again if there is an unknown shift in the bilevel signal then there may be three transitions in an interval $[nT, (n+1)T]$ and so we increase the number of samples by sampling with $\varphi_1(t/(T/2))$. The pseudo-code for sampling bilevel signals using the box and hat functions are given in full detail in [5].

When going to higher order splines, necessity carries over. Sufficiency is more tedious since we must solve a system of higher order polynomial equations.

B. Piecewise polynomials

Similar to bilevel signals we consider sampling piecewise polynomials with the box sampling kernel. Consider an infinite length piecewise polynomial signal $x(t)$ where each piece is a polynomial of degree R and defined on an interval $[t_{k-1}, t_k]$, that is,

$$x(t) = \begin{cases} x_0(t) = \sum_{m=0}^{R} c_{0m}\, t^m & t \in [0, t_0] \\ x_1(t) = \sum_{m=0}^{R} c_{1m}\, t^m & t \in [t_0, t_1] \\ \vdots & \\ x_K(t) = \sum_{m=0}^{R} c_{Km}\, t^m & t \in [t_{K-1}, t_K] \\ \vdots & \end{cases} \quad (125)$$

Each polynomial piece $x_k(t)$ contains $R+1$ unknown coefficients c_{km}. The transition value t_k is easily obtained once the pieces $x_{k-1}(t)$ and $x_k(t)$ are determined, thus there are $2(R+1)+1$ degrees of freedom. If there is one transition in an interval of length T the maximal local rate of innovation is $\rho_m(T) = (2(R+1)+1)/T$. Therefore in order to recover the polynomial pieces and the transition we need to have at least $2(R+1)+1$ samples per interval T. This is achieved by sampling with the following box sampling kernel $\varphi_0(t/\frac{T}{2(R+1)+1})$. For example if $x(t)$ is a piecewise linear signal with 2 pieces as illustrated in Fig. 16 then to recover the

signal it is sufficient to take 5 samples: two before the transition, two after the transition and one sample covering the transition.

We can generalize by noting that the Rth derivative of a piecewise polynomial of degree R is a piecewise constant signal. The pseudo-code for sampling piecewise constant signals with the box sampling kernel is found in [5] .

VII. APPLICATIONS

The applications we consider involve the discrete-time periodic stream of Diracs and piecewise polynomial signals. It is well known that a bandlimited signal can be perfectly recovered from its samples by sampling it at twice the maximum frequency. What if the bandlimited signal has a jump or a discontinuity then the signal is no longer bandlimited and the usual method is not valid. These are what we call piecewise bandlimited signals. Another type of non-bandlimited signal which we may come across in nature is a signal which is obtained from a system with a certain frequency response. The output of the system is a filtered signal. We will look at filtered stream of Diracs and filtered piecewise polynomials.

A. Piecewise bandlimited signals

A discrete-time periodic piecewise bandlimited signal is the sum of a bandlimited signal with a stream of Diracs in the simplest case or with a piecewise polynomial signal. An example is illustrated in Figure 17(e) and is obviously not bandlimited from Figure 17(f). Formally, we have the following

Definition 4: Piecewise bandlimited signals.

Let $\mathbf{x}_{BL}$ be a discrete-time periodic $L-$bandlimited signal of period N with corresponding DTFS coefficients $\mathbf{X}_{BL}$ such that $X_{BL}[m] = 0 \quad \forall m \notin [-L, L]$. Let $\mathbf{x}_{PP}$ be a zero mean discrete-time piecewise polynomial signal of period N with K pieces and with each piece of

maximum degree R. Then a piecewise bandlimited signal $\mathbf{x}$ is defined by

$$\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP} \tag{126}$$

with corresponding DTFS coefficients $\mathbf{X}$ defined by

$$X[m] = \begin{cases} X_{BL}[m] + X_{PP}[m] & \text{if } m \in [-L, L] \\ X_{PP}[m] & \text{if } m \notin [-L, L] \end{cases}. \tag{127}$$

First consider a stream of K weighted Diracs, $\mathbf{x}_{PP}$. From Section III-A, we can recover the K weighted Diracs from $2K$ contiguous frequency values $\mathbf{X}_{PP}$. Since the DTFS coefficients of the bandlimited signal, $\mathbf{X}_{BL}$, are equal to zero outside of the band $[-L, L]$, we have that the DTFS coefficients of the signal outside of the band $[-L, L]$ are exactly equal to the DTFS coefficients of the piecewise polynomial, that is, $\mathbf{X}[m] = \mathbf{X}_{PP}[m], \quad \forall |m| > L$. Therefore it is sufficient to take the $2K$ DTFS coefficients of the outside of the band $[-L, L]$, for instance in $[L+1, L+2K]$. Suppose we have the DTFS of the signal $X[m]$, with $m \in [-(L+2K), L+2K]$ then the DTFS of the bandlimited signal are obtained by subtracting $X_{PP}[m]$ from $X[m]$ for $m \in [-L, L]$.

Recall that the piecewise polynomial has $2K(R+1)$ degrees of freedom and the bandlimited signal has $2L+1$. It follows that we can sample the signal using a discrete-time periodized differentiated sinc sampling kernel bandlimited to $2K(R+1)+L$.

Corollary 1: Consider a piecewise bandlimited signal $\mathbf{x}$ as defined in Definition. 4. Let $\psi[n]$ be the $(R+1)$th differentiated sinc sampling kernel with DTFS

$$\Psi[m] = (D[m])^{R+1}\, Rect_{[-(2K(R+1)+L), 2K(R+1)+L]}. \tag{128}$$

Let M be an integer divisor of N, and let $N/M \geq 2(2K(R+1)+L)$ then the samples

$$y_s[l] = < x[n], \psi[n-lM] >, \quad l = 0, \ldots, N/M-1 \tag{129}$$

are a sufficient representation of $\mathbf{x}$.

Proof: The proof is exactly the same as in Theorem 1 until equation (43)

$$
\begin{aligned}
Y_s[k] &= \frac{1}{M}(1-W_N^{-k})^{R+1}\,X[k], k=0,\ldots,2K(R+1)+L & (130)\\
&= \begin{cases} \frac{1}{M}(1-W_N^{-k})^{R+1}\,(X_{BL}[k]+X_{PP}[k]) & \text{if } k=0,\ldots,L \\ \frac{1}{M}(1-W_N^{-k})^{R+1}\,X_{PP}[k] & \text{if } k=L+1,\ldots,2K(R+1)+L \end{cases} & (131)
\end{aligned}
$$

Therefore $2K(R+1)$ values of

$$X_{PP}[k] = \frac{M}{(1-W_N^{-k})^{R+1}}\,Y_s[k], \quad k \in [L+1, 2K(R+1)+L] \qquad (132)$$

are sufficient to recover the piecewise polynomial $\mathbf{x}_{PP}$. From these we can recover the L spectral components of the bandlimited signal since

$$X_{BL}[k] = \frac{1}{(1-W_N^{-k})^{R+1}}(Y_s[k]-X_{PP}[k]), \quad k=0,\ldots,L. \qquad (133)$$

This gives us the the bandlimited signal $\mathbf{x}_{BL}$ and thus the piecewise bandlimited signal as defined in Definition. 4 is recovered $\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP}$. ∎

Figure 18 the illustrates the reconstruction of a bandlimited plus a piecewise constant signal using the following reconstruction scheme:

Algorithm 2: Reconstruction of piecewise bandlimited signals.

Require: $N, M, N/M \geq 2(2K(R+1)+L)+1$;

Calculate the samples $y_s[l] =< x[n], \psi[n-lM] >$, $l = 0,\ldots,N/M-1$;

Calculate the DTFS $X[m], m \in [-(2K(R+1)+L), (2K(R+1)+L)]$ from the DTFS of samples $y_s[l] \longrightarrow X_{PP}[m] = X[m], m \in [L+1, (2K(R+1)+L)]$;

Solve $\mathbf{h} * X_{PP}[m] = 0, m \in [L+1, (2K(R+1)+L)] \longrightarrow \mathbf{x}_{PP}$;

Calculate $X_{BL}[m] = X[m] - X_{PP}[m], \quad m \in [-L, L] \longrightarrow \mathbf{x}_{BL}$;

The reconstruction is $\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP}$.

B. Filtered piecewise polynomials

Another application of sampling piecewise polynomial signals consists in sampling their filtered output. Figure 19 illustrates that a filtered stream of Diracs is not bandlimited. These signals are formally defined in the following

Definition 5: Filtered piecewise polynomials.

Let $\mathbf{x}_{PP}$ be a zero mean discrete-time periodic piecewise polynomial signal of period N with K pieces of maximum degree R. Let $\mathbf{g}$ be a filter with DTFS $\mathbf{G}$. Then a filtered piecewise polynomial $\mathbf{x}$ is defined by

$$\mathbf{x} = \mathbf{g} * \mathbf{x}_{PP} \tag{134}$$

and the corresponding DTFS coefficients $\mathbf{X}$ are defined by

$$X[m] = G[m] \cdot X_{PP}[m], \quad m = 0, \dots, N-1. \tag{135}$$

Suppose $\mathbf{x}_{PP}$ is a stream of K Diracs. If the filter has $2K$ contiguous nonzero frequency values $G[m]$ then $2K$ frequency values of the signal $X[m]$ will be enough to determine $2K$ frequency values of the stream of Diracs, since $X_{PP}[m] = X[m]/G[m]$, and from Proposition 1 these are sufficient to recover the stream of Diracs .

Corollary 2: Consider a filtered piecewise polynomial signal $\mathbf{x}$ as defined in Definition 5 with $G[m] \neq 0, m \in [-K(R+1), K(R+1)]$. Consider an $(R+1)$ differentiated sinc sampling kernel $\psi[n]$ with DTFS

$$\Psi[m] = (D[m])^{R+1}\, Rect_{[-K(R+1),K(R+1)]}. \tag{136}$$

Let M be an integer divisor of N such that $N/M \geq 2K(R+1)+1$. Then the filtered piecewise polynomial signal can be recovered from the N/M samples

$$y_s[l] = < x[n], \psi[n - lM] >, \quad l = 0, \dots, N/M - 1. \tag{137}$$

Proof: Similar to the proof of piecewise bandlimited signals, we have that the DTFS coefficients of the samples $y_s[l]$ are equal to

$$Y_s[k] = \frac{1}{M}(1 - W_N^{-k})^{R+1} X[k], \quad k \in [-K(R+1), K(R+1)] \tag{138}$$

$$= \frac{1}{M}(1 - W_N^{-k})^{R+1} \left(G[k] X_{PP}[k]\right). \tag{139}$$

Since $G[k] \neq 0$ for $k \in [-K(R+1), K(R+1)]$ we have $2K(R+1)$ values of the DTFS of the piecewise polynomial

$$X_{PP}[k] = \frac{M}{(1 - W_N^{-k})^{R+1} G[k]} Y_s[k], \quad k \in [-K(R+1), K(R+1)] \tag{140}$$

which are sufficient to recover $\mathbf{x}_{PP}$ and which leads to the filtered signal by Definition. 5. ■

The reconstruction scheme is described in the following algorithm and an example of the reconstruction is illustrated in Figure 20.

Algorithm 3: **Require:** $N, M, N/M \geq 2K(R+1)+1$;

Calculate the samples $y_s[l] = < x[n], \psi[n - lM] >, l = 0, \ldots, N/M - 1$;

Calculate $\mathbf{Y}_s = \mathbf{DFT}_{N/M} \cdot \mathbf{y}_s \longrightarrow X[m], m \in [-K(R+1), K(R+1)]$;

Calculate $X_{PP}[m] = X[m]/G[m], m \in [-K(R+1), K(R+1)]$;

Solve $\mathbf{h} * X_{PP}[m] = 0, m \in [-K(R+1), K(R+1)] \longrightarrow \mathbf{x}_{PP}$;

The reconstruction is $\mathbf{x} = \mathbf{g} * \mathbf{x}_{PP}$.

We have seen that the crux of the proof relies on the fact that the filter is known and is invertible over the number of degrees of freedom of the problem. What if the filter has a finite rate of innovation but is unknown? This is more complex and remains to be investigated.

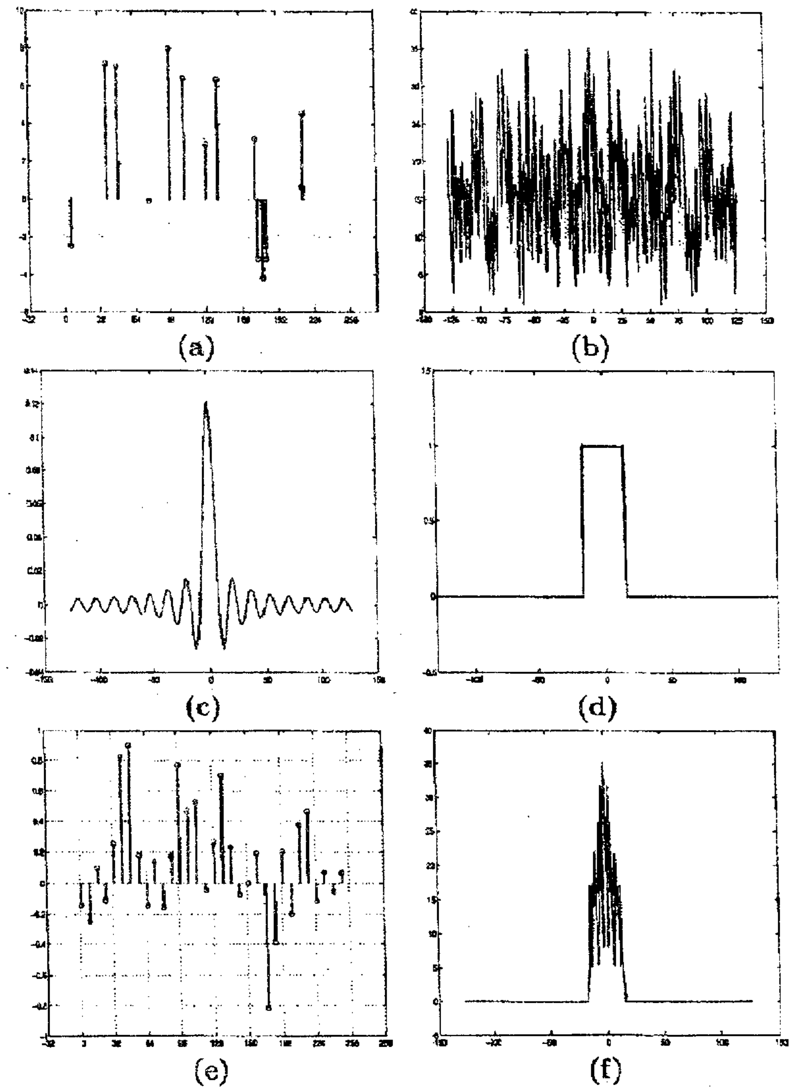


Fig. 3. (a) Periodic discrete-time signal with $K = 15$ weighted Diracs with period $N = 256$;(b) DTFS $X[m]$; (c) Discrete-time periodized sinc sampling kernel, $\varphi[n]$; (d) DTFS $Rect_{[-K,K]}$, $K = 15$(e) Sample values $y_s[l] =< x[n], \varphi[n - lM] >, l = 0, \ldots, 31$ with $M = 8$; (f) DTFS $\mathbf{Y}_s$.

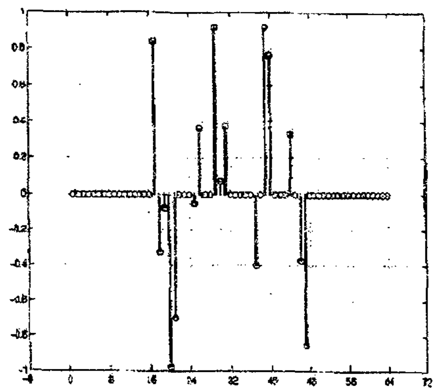


Fig. 4. Stream of $K = 16$ bunched Diracs with period $N = 64$.

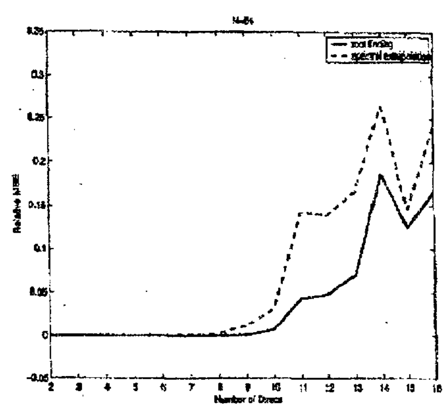


Fig. 5. Comparison between the root finding method and the spectral extrapolation method on a signal of length $N = 64$, K varying between 2 and 16 on interval $2K$, 100 simulations.

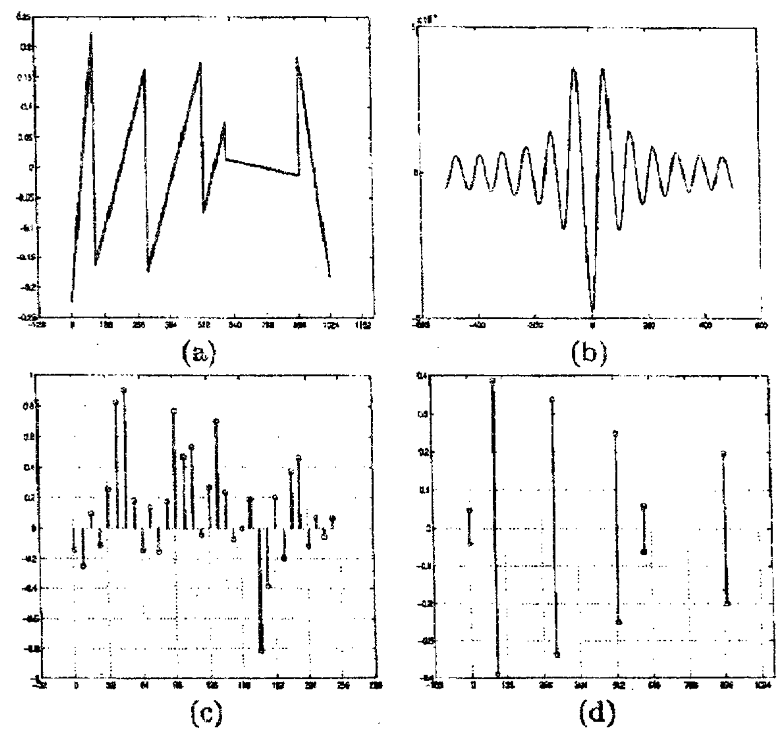
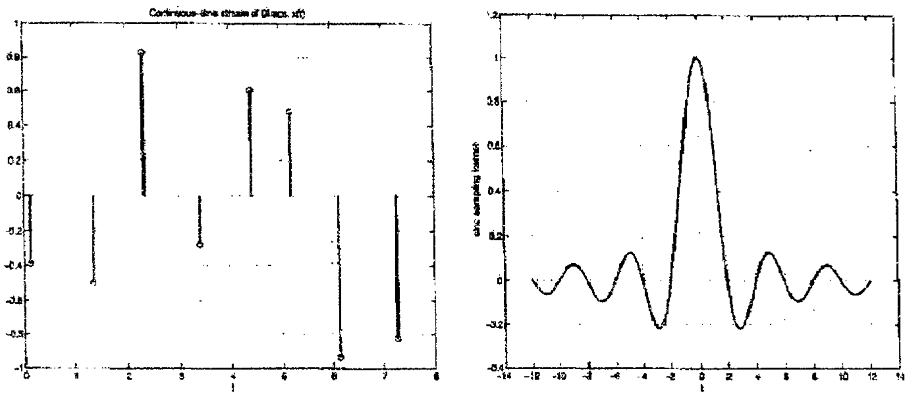
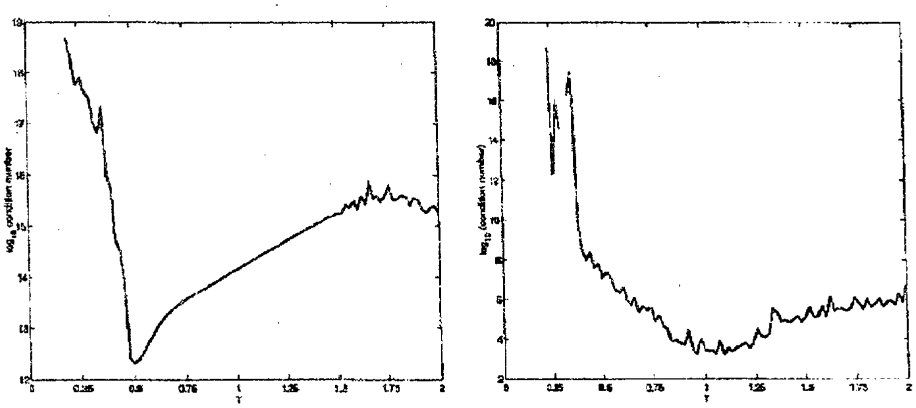


Fig. 6. (a) Discrete-time periodic piecewise linear ($R = 1$) signal of period $N = 1024$ with $K = 6$ pieces; (b) Differentiated sinc sampling kernel, $\psi[n] = d[n] * d[n] * \varphi[n]$ with DTFS $D[m] \cdot Rect_{[-K(R+1),K(R+1)]}$ (c) Sample values $y_s[l] = < x[n], \psi[n - lM] >, l = 0, \ldots, 31$ with $M = 32$; (d) Stream of $K(R+1) = 12$ Diracs obtained from $X[m], m \in [-K(R+1), K(R+1)]$.



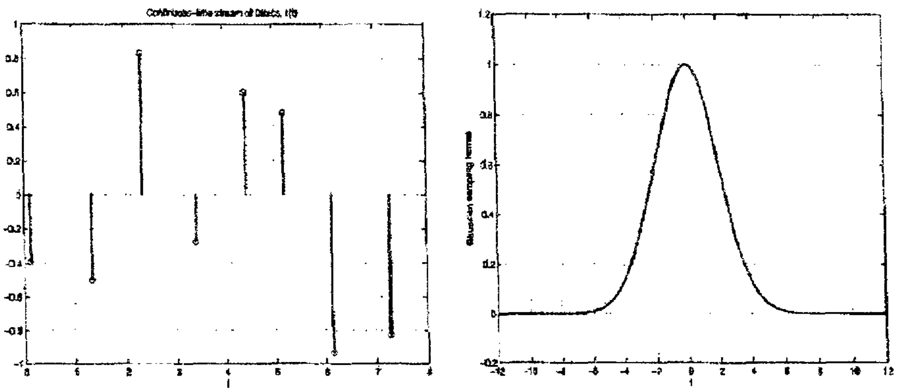
(a) (b)

Fig. 7. (a) Example of a finite length continuous-time stream of $K = 8$ Diracs randomly spread on an interval $[0, \tau]$ with $\tau = 8$; (b) Sinc sampling kernel, $\mathrm{sinc}(t/T), T = 2$.



(a) (b)

Fig. 8. (a) Average condition number of the matrix that leads to the locations of the Diracs, **V**, versus the sampling interval T, optimal $T \approx 0.5$; (b) Average condition number of the matrix that leads to the weights of the Diracs, **A**, versus the sampling interval T, optimal $T \approx 1$. Average is taken on 100 signals with 8 Diracs uniformly spread in the interval $[0, 8]$.



(a) (b)

Fig. 9. (a) Example of a finite length continuous-time stream of $K = 8$ Diracs randomly spread on an interval $[0, \tau]$ with $\tau = 8$; (b) Gaussian sampling kernel, $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}, \sigma = 2$.

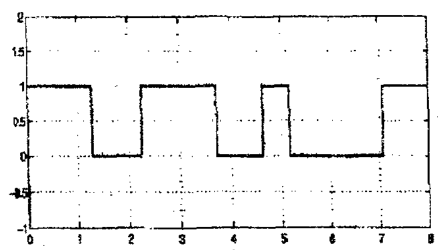
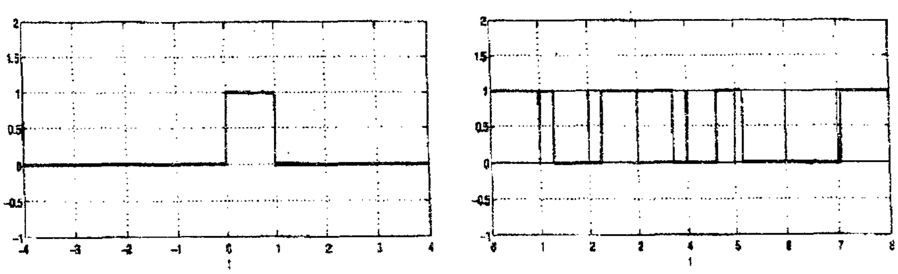


Fig. 10. Bilevel signal.



(a) (b)

Fig. 11. (a) Box spline sampling kernel, $\varphi_0(t)$, $T = 1$. (b) Bilevel signal sampled with the box sampling kernel.

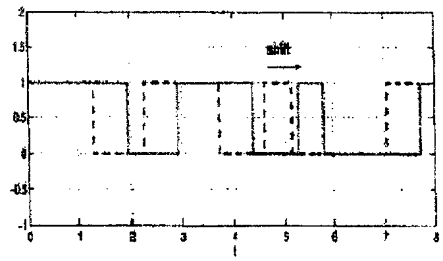


Fig. 12. Shifted bilevel signal with two transitions in the interval [5, 6].

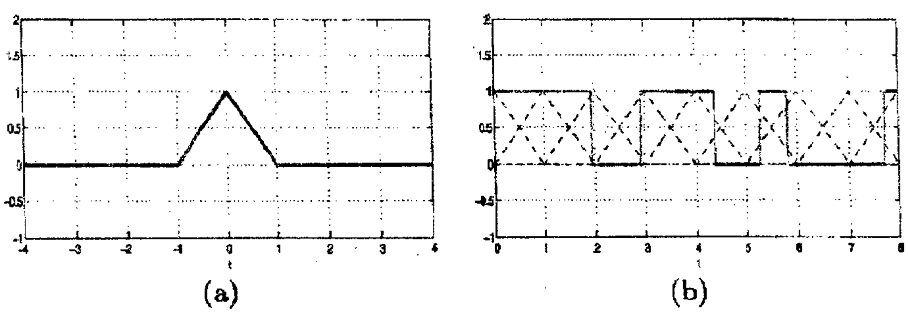


Fig. 13. (a) Hat spline sampling kernel, $\varphi(t/T)$, $T = 1$. (b) Bilevel signal with two transitions in an interval $[n, n+1]$ sampled with a hat sampling kernel.

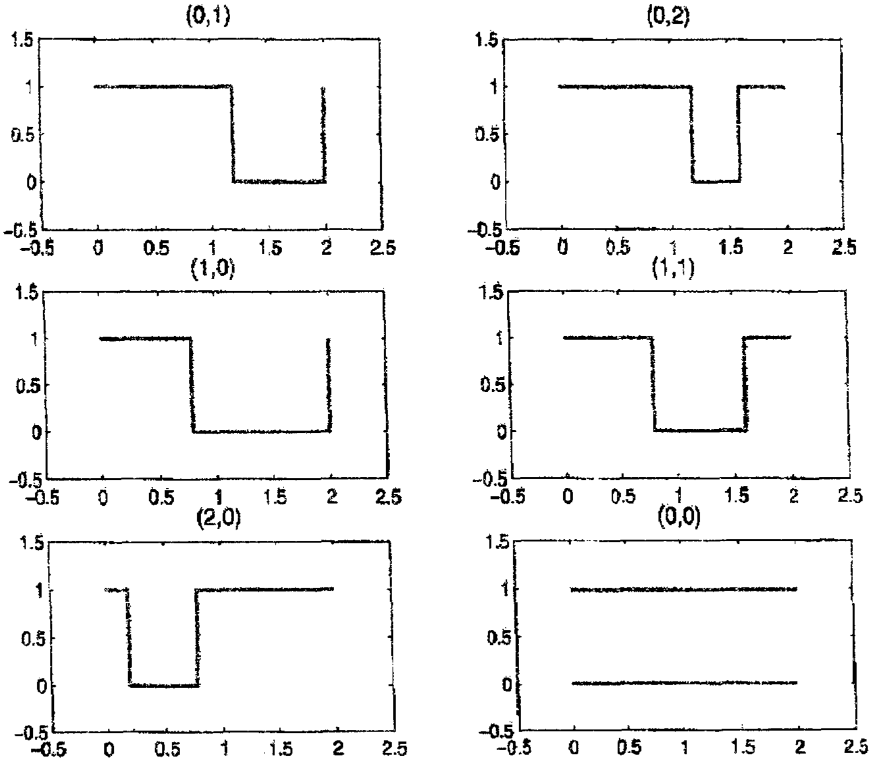


Fig. 14. Bilevel signal containing at most 2 transitions in the interval $[0, 2]$: All possible configurations.

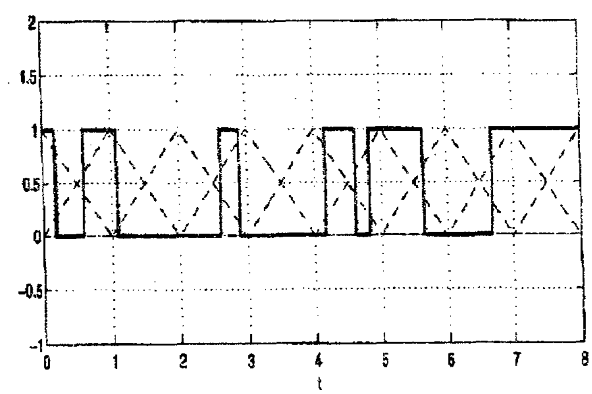


Fig. 15. Bilevel signal containing three transitions in an interval $[4, 5]$, sampled with the hat sampling kernel $\varphi_1(t)$.

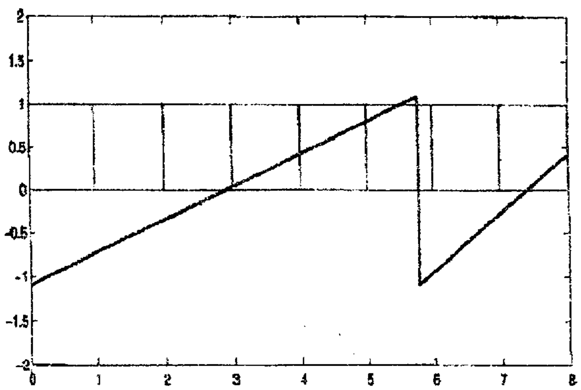


Fig. 16. Piecewise linear signal sampled with a box sampling kernel.

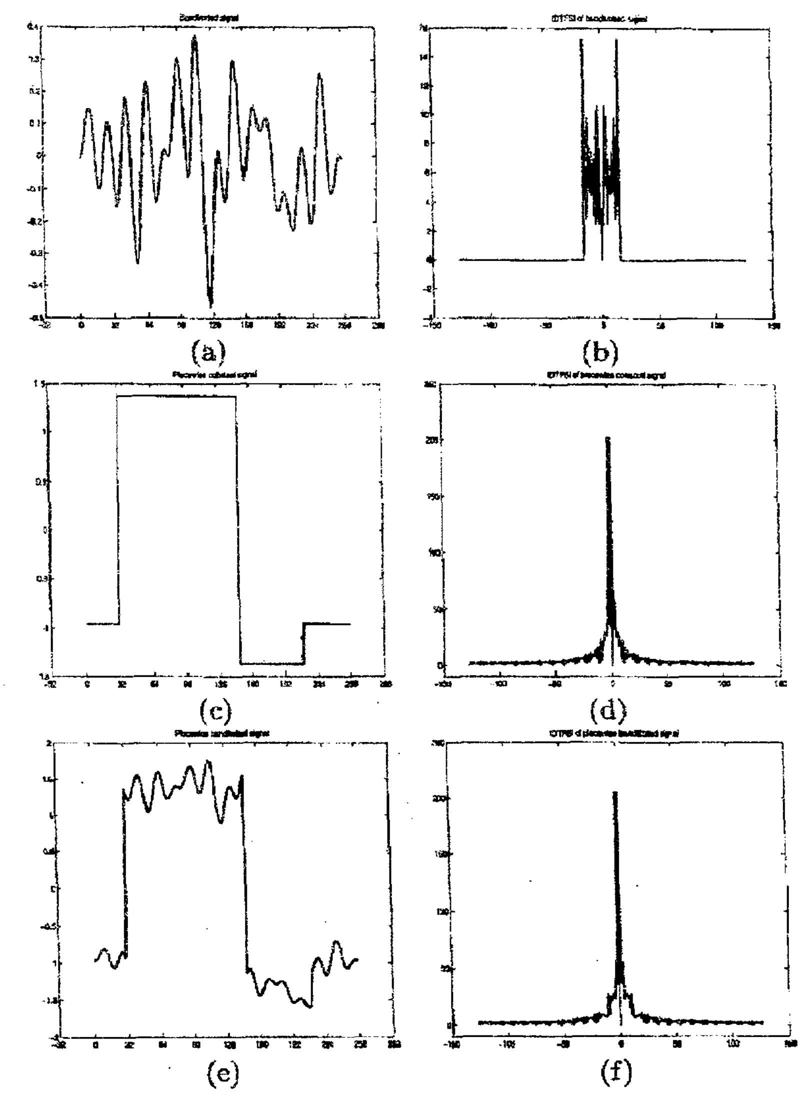


Fig. 17. (a) Bandlimited signal of length $N = 256$; (b) DTFS of Bandlimited signal, $L = 15$ (c) Piecewise constant signal with $K = 3$ pieces; (d) DTFS of piecewise constant signal; (e) Bandlimited piecewise constant signal; (f) DTFS of bandlimited piecewise constant signal.

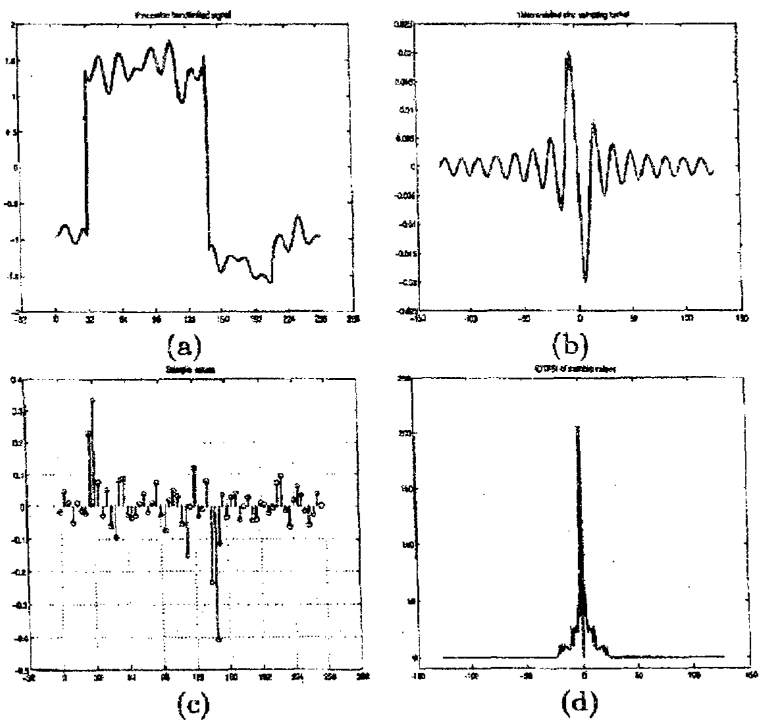


Fig. 18. (a) Bandlimited piecewise constant signal,$x[n]$, with $K = 3$, $L = 15$,$R = 0$,$N = 256$; (b) Differentiated sinc sampling kernel, $\psi[n] = d[n] * \varphi[n]$, bandlimited to $2K(R+1)+1+L = 22$ (c) Sample values $y_s[l] =< x[n], \psi[n-lM] >, l = 0, \ldots, N/M-1$, $M = 4$; (d) $|DTFS|$ of sample values; Reconstruction error is 10^{-13}.

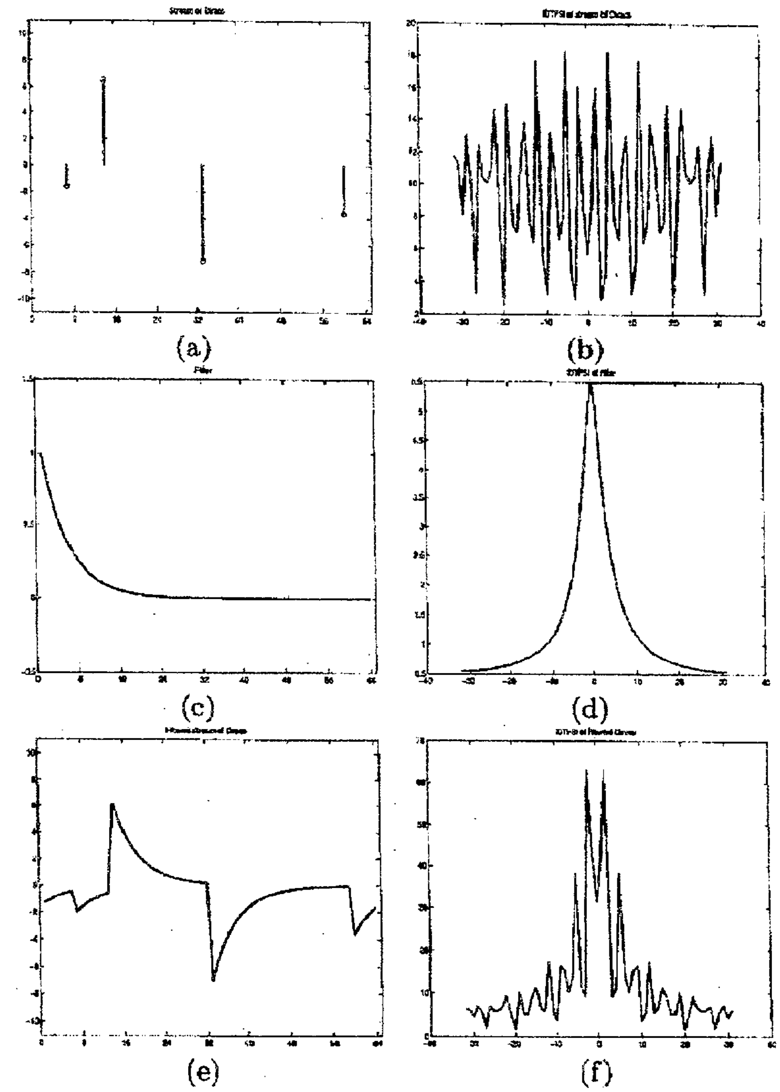


Fig. 19. (a) Stream of $K = 4$ Diracs with period $N = 64$; (b) $|DTFS|$ of stream of Diracs (c) Known filter $g[n] = \alpha^n, n = 0, \ldots, N-1$, $\alpha = 0.4$; (d) $|DTFS|$ of filter; (e) Filtered stream of Diracs; (f) $|DTFS|$ of filtered stream of Diracs.

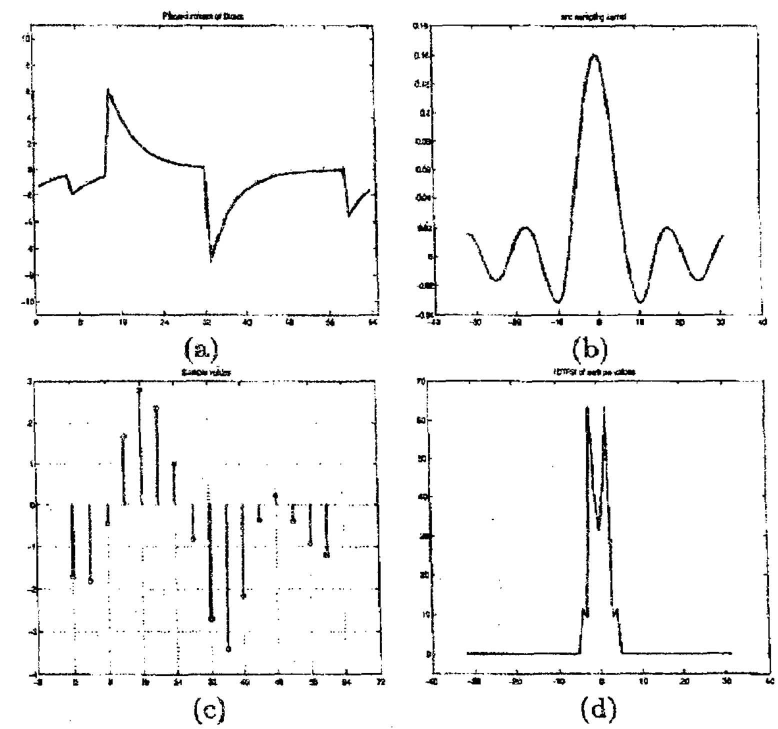


Fig. 20. (a) Filtered stream of Diracs, $x[n]$, $N = 64$; (b) Sinc sampling kernel $\varphi[n]$ bandlimited to $K = 4$ (c) Sample values $y_s[l] = < x[n], \varphi[n - lM] >, l = 0, \ldots, 15$, $M = 4$; (d) $|DTFS|$ of sample values; Reconstruction error is 10^{-13}.

VIII. Conclusion

- We derived sampling theorems for periodic signals in particular streams of weighted Diracs and piecewise polynomials. These signals have a finite rate of innovation ρ which is equal to the number of degrees of freedom per period.
- The samples are obtained by taking the inner product of the signal with a shifted version of the periodized sinc kernel or differentiated sinc kernels. The bandwidth of these kernels must be greater or equal to the degrees of freedom of the signal.
- The discrete-time periodic signals are perfectly recovered when the sampling rate $1/M$ is greater or equal to the rate of innovation $\rho = 2K/N$ in the case of streams of weighted Diracs or $\rho = 2K(R+1)/N$ in the case of a piecewise polynomial signal with K pieces and maximum degree R.
- The continuous-time periodic streams of Diracs and piecewise polynomial signals are perfectly recovered when the sampling rate $1/T$ is greater or equal to the rate of innovation $\rho = 2K/\tau$ since we assumed that the piecewise polynomial signal belonged to C^{R-1}.
- The sampling and reconstruction scheme is illustrated in Figure 21.

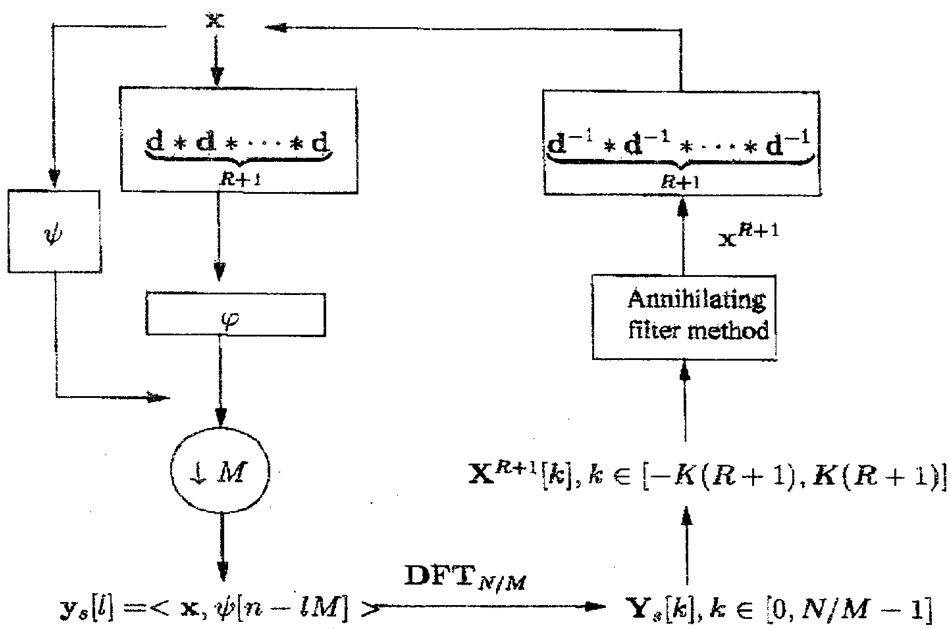


Fig. 21. Sampling and reconstruction scheme for discrete-time piecewise polynomial signals with K pieces of maximum degree R; N/M is the number of samples; $2K(R+1)+1$ is the bandwidth of the sampling kernel.

- A finite length stream of K Diracs can be recovered from N samples y_n obtained as the inner product between the signal and shifted versions of the sinc and Gaussian sampling kernel, when $N \geq 2K$.
- For both types of sampling kernels two systems of equations must be solved: the first system is to find the locations of the Diracs and the second is to find the weights of the Diracs.
- When sampling a randomly spaced stream of Diracs with the *sinc* kernel the system leading to the transitions may be ill-conditioned if the sampling interval T is not chosen appropriately. It is illustrated that at critical sampling, that is, when we have $N = 2K$ sample values, the optimal sampling interval obtained for these type of signals is $T = 0.5$.
- When sampling a randomly spread stream of Diracs with the *Gaussian* kernel the conditioning of both systems depends also on the value of the variance σ^2 in the Gaussian kernel.
- The sampling schemes using the sinc and the Gaussian kernels can be generalized to both continuous-time and discrete-time piecewise polynomial signals.
- Infinite length signals were sampled using a compact support sampling kernel.
- Bilevel signals can be recovered using a *Box* sampling kernel $\varphi_0(t/T)$ if and only if there is at most one transition in each interval $[n, (n+1)T]$.
- Bilevel signals can be recovered using a *Hat* sampling kernel $\varphi_1(t/T)$ if and only if there is at most two transitions in each interval $[n, (n+2)T]$.
- In general, to recover the infinite length piecewise polynomials with K pieces of of maximum degree R using a box sampling kernel, the sampling rate must be greater

than the maximum local rate of innovation

$$\rho_m(T) = (2(R+1)+1)/T.$$

- Sampling and reconstruction algorithms were given for each problem in their respective sections.

Appendix

I. Annihilating filter method

The problem in spectral line analysis consists in estimating the frequencies of a sinusoidal signal from a set of values. The methods used for estimating such frequencies are known as high-resolution methods, for example MUSIC, ESPRIT and can be found in [7]. We define the following as the annihilating filter method:

Consider a signal $s[n], n \in \mathbb{Z}$ defined as a finite lineat combination of K exponentials u_k^n,

$$s[n] = \sum_{k=0}^{K-1} c_k\, u_k^n \tag{141}$$

where c_k are real and u_k are real or complex valued. In the context of spectral line analysis $u_k = e^{i\omega_k}$ where ω_k is the kth frequency component of the signal $s[n]$.

Definition 6: A filter $1 - z^{-1}u_k$ is called an annihilating filter for u_k^n if [7]

$$(1 - z^{-1}u_k)u_k^n \equiv 0. \tag{142}$$

where z^{-1} is a shift or delay operator.

Suppose that $u_k \neq 0$, then $z = u_k$ satisfies (142) and is a zero of the filter $1 - z^{-1}\, u_k$. If there are K exponentials then there are K filters $1 - z^{-1}u_k$ each annihilating their respective u_k^n. This implies that the product of these filters

$$H(z) = \prod_{l=0}^{K-1}(1 - z^{-1}u_l) \tag{143}$$

for sure annihilates each exponentials, u_k^n, thus $H(z)$ is an annihilating filter of $s[n]$,

$$s[n] \cdot H(z) \equiv 0. \tag{144}$$

Therefore to find the values u_k we need to find the filter coefficients h_k in

$$H(z) = \sum_{m=0}^{K} h_k\, z^{-k} \tag{145}$$

such that (144) is satisfied. Substituting $H(z)$ defined in (145) in (144) we obtain

$$\sum_{k=0}^{K} h_k \, s[n] \, z^{-k} = 0, \quad n \in \mathbb{Z} \tag{146}$$

which equivalent to the following recurrence equation

$$\begin{aligned} \sum_{k=0}^{K} h_k \, s[n-k] &= 0 && (147) \\ \mathbf{h} * \mathbf{s} &= 0. && (148) \end{aligned}$$

In matrix/vector form the system in (147) is equivalent to

$$\begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ s[0] & s[-1] & \cdots & s[-K] \\ s[1] & s[0] & \cdots & s[-(K-1)] \\ \vdots & \vdots & \ddots & \\ s[K] & s[K-1] & \cdots & s[0] \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \cdot \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_K \end{pmatrix} = 0. \tag{149}$$

Suppose a finite number of values $s[n]$ are available. Since there are $K+1$ unknown filter coefficients we need at least $K+1$ equations, and therefore we need at least $2K+1$ values of $s[n]$ to find the filter coefficients.[8] Define $\mathbf{S}$ the appropriate sub-matrix then the system $\mathbf{S} \cdot \mathbf{h} = 0$ will admit a solution when

$$rank(\mathbf{S}) < K+1. \tag{150}$$

Once the the filter coefficients are found then the values u_k are simply the roots of the annihilating filter $H(z)$.

To determine the weights c_k it suffices to take K equations in (141) and solve the system for c_k. Let $n = 0, \dots, K-1$ then in matrix vector form we have the following Vandermonde

[8] Actually there are K unknown filter coefficients since $h_0 = 1$ and therefore we will need at least $2K$ values of $s[n]$. The system to solve in this case is known as a Yule-Walker system [2]

system

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ u_0 & u_1 & \dots & u_{K-1} \\ \vdots & \vdots & \dots & \vdots \\ u_0^{(K-1)} & u_1^{(K-1)} & \dots & u_{K-1}^{(K-1)} \end{bmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{pmatrix} = \begin{pmatrix} s[0] \\ s[1] \\ \vdots \\ s[K-1] \end{pmatrix} \tag{151}$$

and has a unique solution when

$$u_k \neq u_l, \forall k \neq l. \tag{152}$$

This concludes the annihilating filter method.

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List of Figures

Fig. 1. Sampling set up: $x(t)$ is the continuous-time signal; $\varphi(-t)$ is the sampling kernel; T is the sampling interval; $x[n] = < x(t), \varphi(t - nT) >, n = 0, \ldots, N-1$ are the sample values.

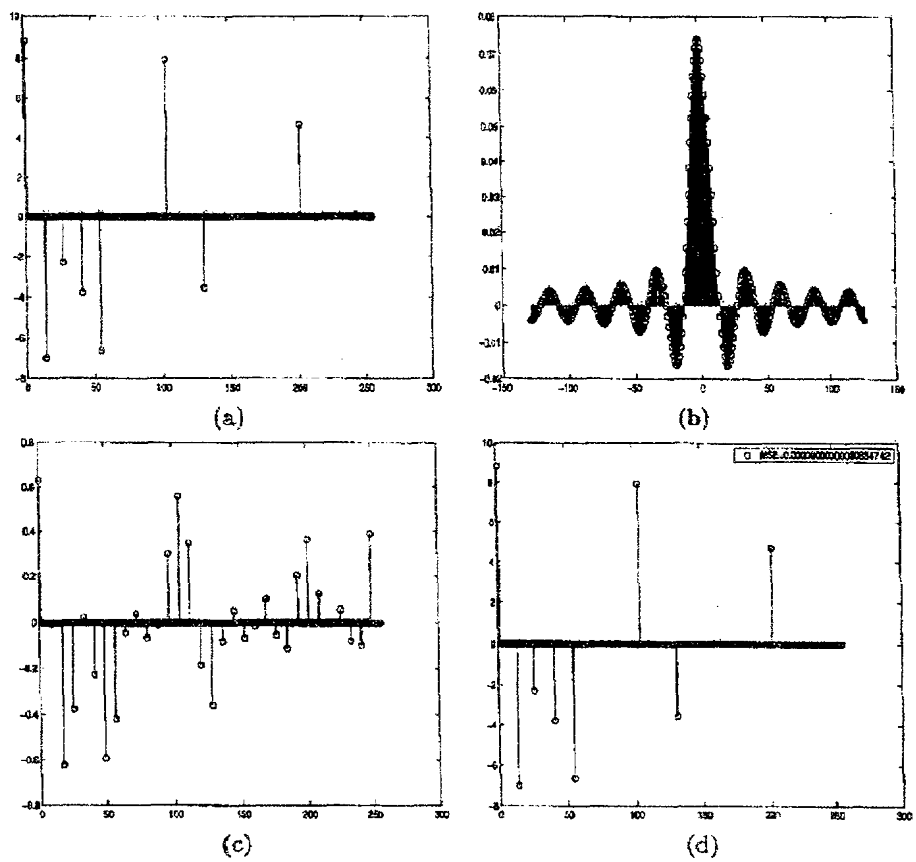
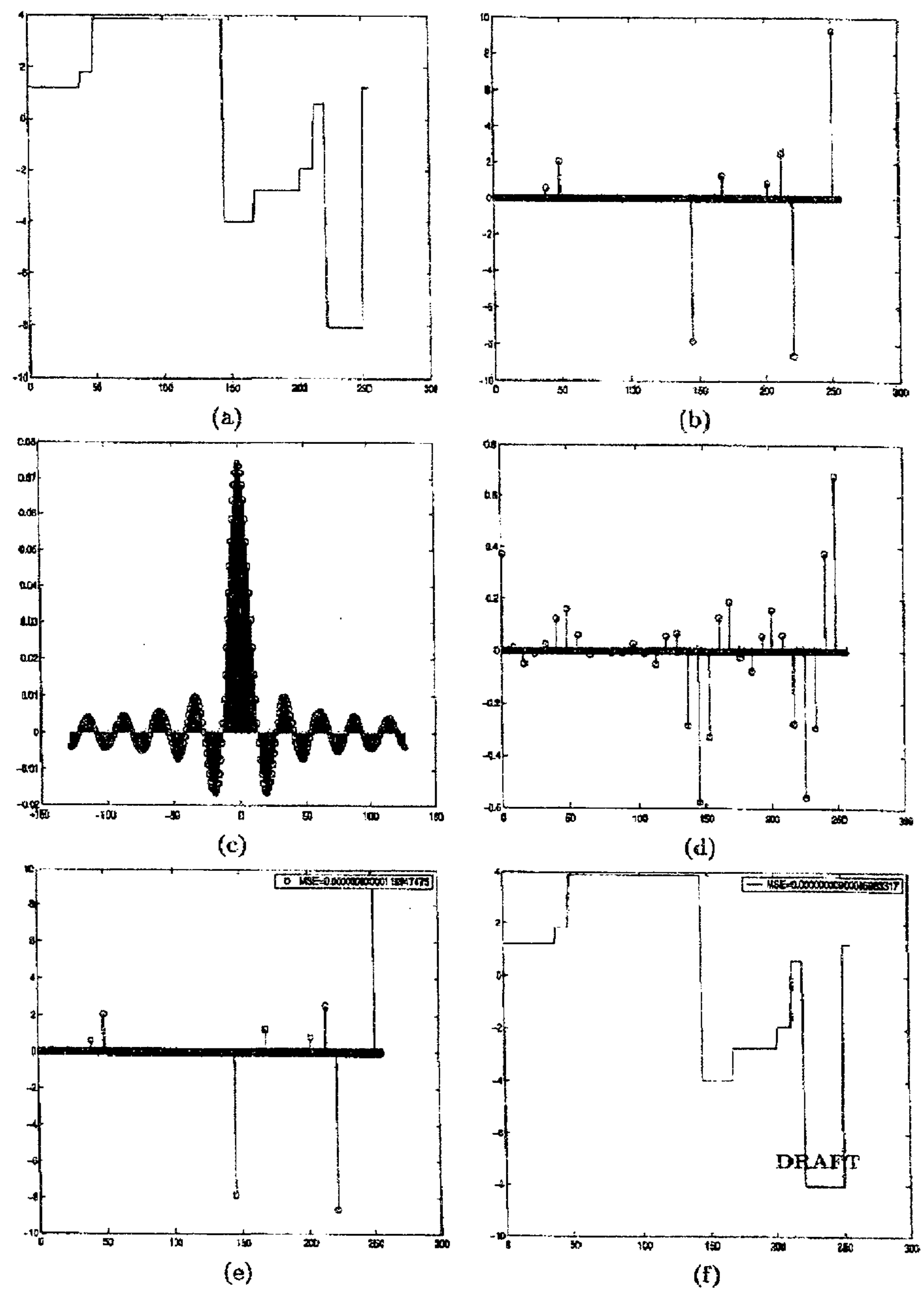


FIGURE 2.



(a) (b) (c) (d) (e) (f)

FIGURE 3

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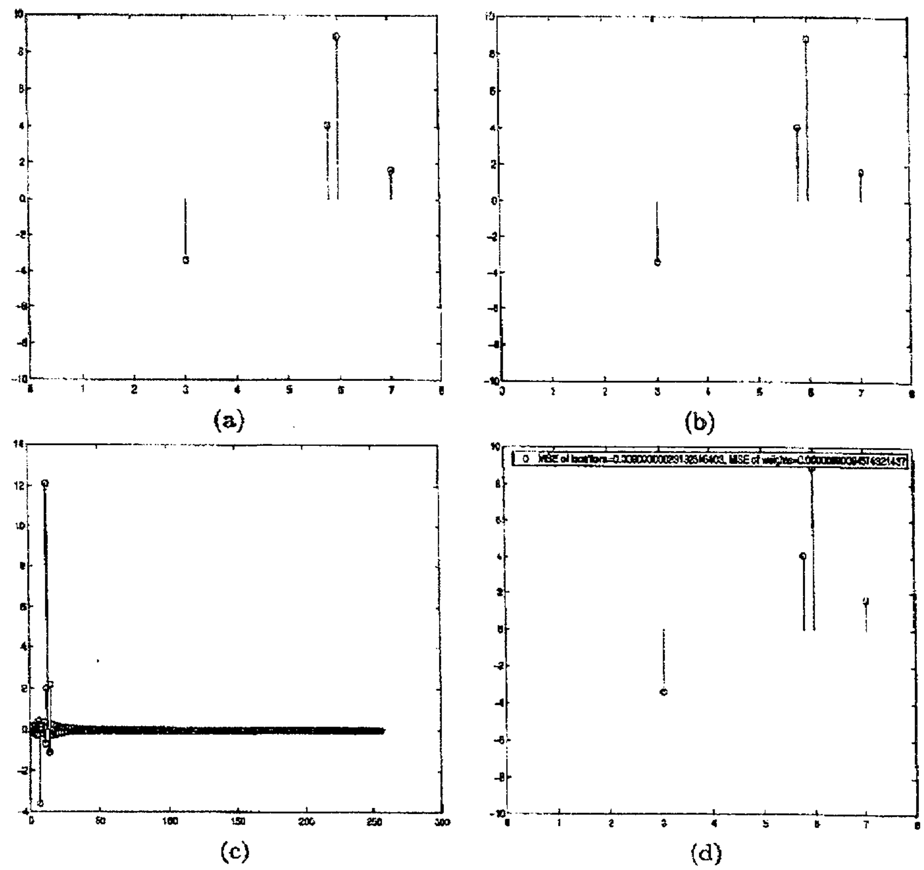
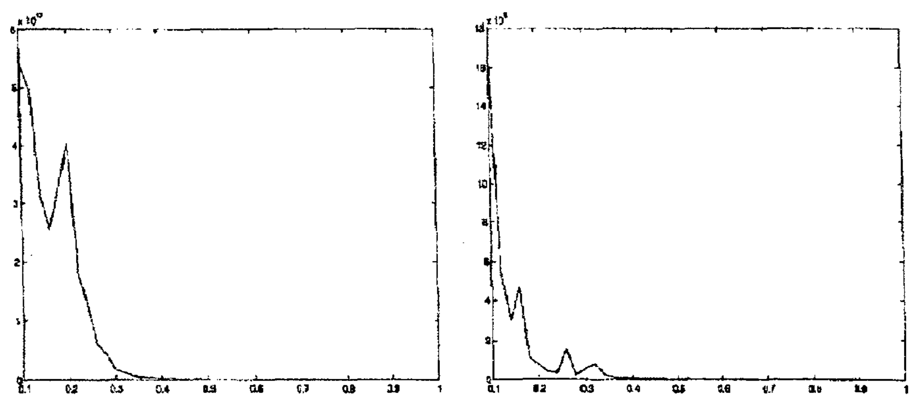


FIGURE 4



(a) (b)

FIGURE 5.

Fig. 6. (a) Finite length continuous-time signal with $K = 8$ weighted Diracs, $x(t), t \in [0, \tau]$, $\tau = 8$; (b) Continuous-time Gaussian sampling kernel, $\varphi(t) = e^{\frac{-t^2}{2\sigma^2}}$; (c) Sample values $y_n = < x(t), e^{\frac{-(\frac{t}{T}-n)^2}{2\sigma^2}} >, n = 0, \ldots, N-1$; (d) Reconstructed finite length continuous-time stream of Diracs, $MSE = 10^{-??}$.

Fig. 7. (a) Conditioning; (b) Reconstruction error versus conditioning.

Fig. 8. (a) Bilevel signal with at most 1 transition in each interval $[n, n+1]$; (b) Box sampling kernel, $\varphi_0(t)$; (c) Sample values $x[n] = < x(t), \varphi_0(t - nT) >$ with $T = 1$.

Fig. 9. (a) Bilevel signal with at most 2 transitions in each interval $[n, n+2]$; (b) Hat sampling kernel, $\varphi_1(t)$; (c) Sample values $x[n] = < x(t), \varphi_1(t - nT) >$ with $T = 1$.

Fig. 10. Non intersecting solution spaces for bilevel signals with hat sampling kernel

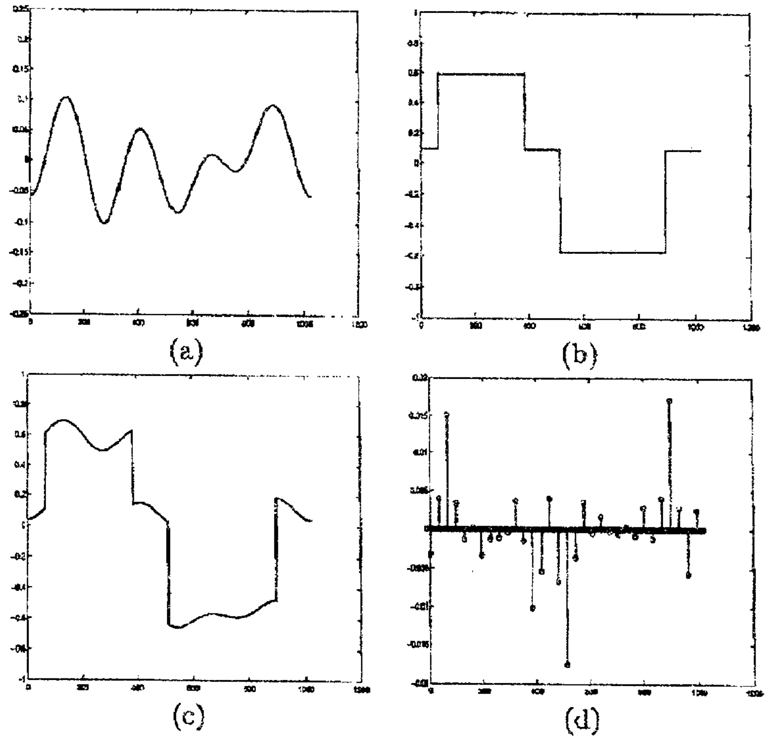


Fig. 11. (a) Periodic discrete-time bandlimited signal $\mathbf{x}_{BL}$ of length $N = 1024$ with $M_{BL} = 4$; (b) Periodic discrete-time piecewise constant signal, $\mathbf{x}_{PW}$, with $K = 4$ transitions; (c) Periodic discrete-time piecewise constant bandlimited signal, $\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PW}$ with $K = 4$ and M=16; (d) Sample values $y_l = < x[n], \psi[n - lT] >, l = 0, \ldots, N/T - 1$, $T = 32$, where $\psi[n] = \varphi[n] - \varphi[n-1]$;(e) Reconstructed piecewise constant signal, $MSE = 10^{-??}$; (f) Reconstructed piecewise constant bandlimited signal, $MSE = 10^{-??}$.

Fig. 12. (a) Discrete-time stream of $K = 4$ Diracs, $\mathbf{x}_{PW}$; (b) Known filter $g[n] = \alpha^n u[n]$, $\alpha = 0.4$ (c) Filtered stream of Diracs, $\mathbf{x} = \mathbf{g} * \mathbf{x}_{PW}$; (d) $y_l = < x[n], \phi[n - lT] >, l = 0, \ldots, 31$, $T = 32$;(e) Reconstructed stream of Diracs, $MSE = 10^{-??}$; (f) Reconstructed filtered stream of Diracs $\mathbf{x}$, $MSE = 10^{-??}$.

Fig. 13. General reconstruction scheme

Chapter 2

Sampling periodic signals with finite rate of innovation

Sampling theory has been extensively developed for bandlimited signals. In this chapter[1] non-bandlimited signals are investigated in particular periodic signals with a finite rate of innovation. Recall that in Section 1.1 signals with a finite rate of innovation ρ are characterized by having a finite number of degrees of freedom per unit of time. For example take a periodic signal of period N with Diracs at K locations. This signal is not bandlimited and has K degrees of freedom in an interval of length N thus its rate of innovation is $\rho = K/N$.
In Sections 2.1 and 2.2 of this chapter, sampling theorems for discrete-time and continuous-time periodic streams of weighted Diracs and piecewise polynomial signals are derived. Both of these type of signals are not bandlimited and have a finite number of degrees of freedom per period. By taking an appropriate sampling kernel and a sufficiently high sampling rate that captures these degrees of freedom, the signals can be perfectly reconstructed. Section 2.3 derives applications of the above results, in particular to piecewise bandlimited signals, and to filtered piecewise polynomials. In all of the proofs of the aforementioned sampling theorems a method that is commonly used in spectral analysis is employed, namely the "annihilating filter method" [67]. For those unfamiliar with this method it is described in Appendix 2.A.

2.1 Discrete-time periodic signals

The discrete-time periodic signals we consider are streams of weighted Diracs and piecewise polynomials. Through appropriate differentiation, piecewise polynomials can be reduced to streams of Diracs, so we begin with these.

2.1.1 Stream of Diracs

Consider a discrete-time periodic signal, with one period given by

$$\mathbf{x} = (x[0], x[1], \ldots, x[N-1])^T \tag{2.1}$$

[1]This chapter includes research conducted jointly with Martin Vetterli and Thierry Blu [80, 79, 78].

and containing K weighted Diracs at locations $\{n_0, n_1, \ldots, n_{K-1}\}$, $n_k \in [0, N-1]$ and $K < \lfloor N/2 \rfloor$,

$$x[n] \quad = \quad \sum_{k=0}^{K-1} c_k \, \delta[n - n_k], \tag{2.2}$$

where $\delta[n]$ is the Kronecker delta and equal to 1 if $n = 0$ and 0 if $n \neq 0$.
Denote by $\mathbf{X} = (X[0], X[1], \ldots, X[N-1])^T$ the discrete-time Fourier series (DTFS) coefficients of $\mathbf{x}$ where

$$X[m] \quad = \quad \sum_{k=0}^{K-1} c_k \, W_N^{n_k m}, \quad m = 0, \ldots, N-1 \tag{2.3}$$

and $W_N = e^{-i2\pi/N}$.
Consider filtering the signal $x[n]$ with a lowpass filter $\tilde{\varphi}[n] = \varphi[-n]$ with bandwidth $[-K, K]$ then the sample values $y_s[l]$ are simply a subsampled version (by M) of the filtered signal $y[n] = x[n] * \tilde{\varphi}[n]$. The DTFS coefficients of $y[n]$ are given by

$$Y[m] = \begin{cases} X[m] & \text{if } m \in [-K, K] \\ 0 & \text{else} \end{cases} \tag{2.4}$$

and those of the subsampled signal $y_s[l] = y[lM]$ are given by the usual subsampling formula

$$Y_s[m] = \frac{1}{M} \sum_{l=0}^{M-1} Y[(m + lN)/M]. \tag{2.5}$$

With appropriate re-indexing it follows that

$$Y_s[m] = \frac{1}{M} X[m], \quad m \in [-K, K]. \tag{2.6}$$

Figure 2.1 illustrates that we can recover $2K$ spectral values $X[m]$ of the original signal from the subsampled spectra of the lowpass approximation $Y_s[m]$ as long as there is no overlapping in the spectra of the lowpass approximation $Y[m]$ and this occurs only if $N/M \geq 2K$. This leads us to

Proposition 2.1 *Consider a discrete-time periodic signal $x[n]$ of period N containing K weighted Diracs. Let M be an integer divisor of N satisfying $N/M \geq 2K + 1$. Consider the discrete-time periodized sinc sampling kernel $\varphi[n] = \frac{1}{N} \sum_{m=-K}^{K} W_N^{-mn}$, that is, the inverse DTFS of the $Rect_{[-K,K]}$. Then the $N/M \in \mathbb{N}$ samples defined by*

$$y_s[l] \quad = \quad < x[n], \varphi[n - lM] >_{circ}, \quad l = 0, \ldots, N/M - 1 \tag{2.7}$$

are a sufficient representation of the signal.

Proof: We start by showing that the DTFS coefficients $X[m], m \in [-K, K]$ are sufficient to determine the stream of K weighted Diracs . Then we show that the N/M samples $y_s[l]$ are a sufficient representation of $X[m], m \in [-K, K]$.

1. Since $X[m]$ is a linear combination of K complex exponentials, u_k^m, with $u_k = W_N^{n_k}$, the locations n_k of the Diracs can be found using the annihilating filter method described in Appendix 2.A. It suffices to determine the annihilating filter $H(z)$ whose coefficients are $(1, H[1], \ldots, H[K])$ or

$$H(z) = 1 + H[1]\, z^{-1} + H[2]\, z^{-2} + \cdots + H[K]\, z^{-K} \tag{2.8}$$

which factors as

$$H(z) = \prod_{k=0}^{K-1} (1 - z^{-1} W_N^{n_k}) \tag{2.9}$$

and satisfies

$$\sum_{k=0}^{K} H[k]\, X[m-k] = 0, \quad m = 0, \ldots, N-1 \tag{2.10}$$

Since $H[0] = 1$, K equations (2.10) will be sufficient to determine the K unknown filter coefficients $H[k], k = 1, \ldots, K$. Let $m = 1, \ldots, K$ then the system in (2.10) is equivalent to

$$\sum_{k=1}^{K} H[k]\, X[m-k] = -X[m], \quad m = 1, \ldots, K. \tag{2.11}$$

For example take $N = 8$, $K = 3$ and let $m = 1, 2, 3$ then in matrix/vector form the system is

$$\begin{bmatrix} X[0] & X[-1] & X[-2] \\ X[1] & X[0] & X[-1] \\ X[2] & X[1] & X[0] \end{bmatrix} \cdot \begin{pmatrix} H[1] \\ H[2] \\ H[3] \end{pmatrix} = -\begin{pmatrix} X[1] \\ X[2] \\ X[3] \end{pmatrix}. \tag{2.12}$$

Given that these are K sinusoids the matrix in (2.12) is full rank ($= K$) and thus there is a unique solution $H[1], \ldots, H[K]$. The set of locations $\{n_0, n_1, \ldots, n_{K-1}\}$ are given by the the zeros of $H(z)$.
The weights of the Diracs are obtained by solving K equations in (2.3), let $m = 0, \ldots, K-1$, this leads to the following Vandermonde system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ W_N^{n_0} & W_N^{n_1} & \cdots & W_N^{n_{K-1}} \\ \vdots & \vdots & \cdots & \vdots \\ W_N^{n_0(K-1)} & W_N^{n_1(K-1)} & \cdots & W_N^{n_{K-1}(K-1)} \end{bmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[K-1] \end{pmatrix} \tag{2.13}$$

and has a unique solution since the $n_k \neq n_l, \forall k \neq l$.
Therefore, given $2K$ contiguous DTFS coefficients

$$\{X[-K+1], X[-K+2], \ldots, X[0], \ldots, X[K]\}$$

we have found a unique set of locations $\{n_k\}_{k=0}^{K-1}$ and a unique set of weights $\{c_k\}_{k=0}^{K-1}$.

2. We need to show that $2K$ spectral values $X[m], m \in [-K, K]$ can be obtained from the N/M sample values $y_s[l]$ defined in (2.7).
We substitute the discrete-time periodized sinc kernel in the expression of the sample values and we obtain the following:

$$
\begin{aligned}
y_s[l] &= \langle x[n], \varphi[n - lM] \rangle_{circ} \quad l = 0, \dots, N/M - 1 && (2.14)\\
&= \sum_{n=0}^{N-1} x[n]\,\varphi[n - lM] && (2.15)\\
&= \frac{1}{N}\sum_{n=0}^{N-1} x[n] \sum_{m=-K}^{K} W_N^{-m(n-lM)} && (2.16)\\
&= \frac{1}{N}\sum_{n=0}^{N-1} x[n] \sum_{m=-K}^{K} W_N^{-mn}\, W_N^{mlM} && (2.17)\\
&= \frac{1}{N}\sum_{m=-K}^{K} W_{N/M}^{ml} \underbrace{\sum_{n=0}^{N-1} x[n]\, W_N^{-nm}}_{X[-m]} && (2.18)\\
&= \frac{1}{N}\sum_{m=-K}^{K} X[-m]\, W_{N/M}^{ml}. && (2.19)
\end{aligned}
$$

If we calculate the DTFS coefficients of the sample values $y_s[l]$ we obtain an expression in terms of the DTFS of the signal,

$$
\begin{aligned}
Y_s[k] &= \sum_{l=0}^{N/M-1} y_s[l]\, W_{N/M}^{lk}, \quad k = 0, \dots, N/M - 1 && (2.20)\\
&= \frac{1}{N}\sum_{l=0}^{N/M-1} \sum_{m=-K}^{K} X[-m]\, W_{N/M}^{ml} W_{N/M}^{lk} && (2.21)\\
&= \frac{1}{N}\sum_{m=-K}^{K} X[-m] \underbrace{\sum_{l=0}^{N/M-1} W_{N/M}^{l(k+m)}}_{=\begin{cases} N/M & \text{if } k+m=0 \\ 0 & \text{otherwise}\end{cases}} && (2.22)\\
&= \frac{1}{M} X[k], \quad k = 0, \dots, \min\{K, N/M - 1\} && (2.23)\\
\Rightarrow X[k] &= M Y_s[k], \quad k = 0, \dots, K && (2.24)
\end{aligned}
$$

by hypothesis, $N/M \geq 2K + 1 > K$. Since we are dealing with real signals the DTFS is Hermitian, that is, $X[-k] = X^*[k]$, $k = 0, \dots, K$, so we have the $2K + 1$ spectral values $X[k]$, $k \in [-K, K]$ obtained from the N/M DTFS coefficients of the sample values $y_s[l]$. Therefore we have a sufficient number of spectral values which uniquely define the stream of weighted Diracs.

■

Figure 2.2 illustrates in time and frequency domain the sampling of a discrete-time periodic stream of Diracs with period $N = 256$ and $K = 15$ weighted Diracs. The signal is perfectly reconstructed within machine precision, $MSE = 10^{-11}$.

Note that in the proof of Prop. 2.1 the locations of the Diracs are determined by finding the roots of the annihilating filter $H(z)$. If the locations are bunched up or there are a large number of Diracs then finding the roots of the polynomial is numerically unstable. An alternative method that is commonly used in error correction coding involves extrapolating the $N - K$ spectral values of the signal using K first spectral $X[k], k = 1, \ldots, K$ components and the error locating polynomial which in our case corresponds to the annihilating filter $H[k], k = 1, \ldots, K$,

$$X[k] = -\sum_{l=1}^{K} H[l]\, X[k-l], \quad k = K+1, \ldots, N-K. \tag{2.25}$$

Consider a signal of length $N = 64$ where there are $K = 16$ Diracs in an interval of size $2K$, see Figure 2.3. Figure 2.4 compares the relative reconstruction error between the root finding method and the spectral extrapolation method for different values of K.

2.1.2 Piecewise polynomials of degree R

The previous result on the stream of Diracs is extended to piecewise polynomials. Consider a discrete-time periodic piecewise polynomial defined by [2]

$$x[n] = \frac{1}{R!} \sum_{k=0} c_k \, (n - n_k)_+^R \tag{2.26}$$

of period N with K pieces each with maximum degree R. Suppose a discrete-time difference operator $d[n] = \delta[n] - \delta[n-1]$ is applied $R+1$ times to the piecewise polynomial signal. The differentiated signal $x^{R+1}[n]$ in frequency domain is

$$X^{(R+1)}[m] = (D[m])^{R+1}\, X[m], \quad m = 0, \ldots, N-1 \tag{2.27}$$

where $D[m] = 1 - W_N^m$ is the DTFS of the discrete-time difference operator. This results in putting to zero all the polynomial pieces. Assume there are discontinuities between pieces (but no Diracs), then K transitions can lead to at most $K(R+1)$ weighted Diracs and thus the rate of innovation is $\rho = 2K(R+1)/N$. From Proposition 2.1 we can uniquely recover the $K(R+1)$ Diracs from $2K(R+1)$ DTFS coefficients of the differentiated signal $X^{(R+1)}[k]$. The piecewise polynomial signal is reconstructed by applying the inverse discrete-time difference operator $R+1$ times on the stream of weighted Diracs. The discrete-time difference operator $d[n]$ is a singular operator (since $D[0] = 0$) and so we define the inverse discrete-time difference operator as $D^{-1}[m] = 0$ for $m = 0$ and $D^{-1}[m] = (1 - W_N^m)^{-1}$ for $m = 1, \ldots, N-1$. Hence instead of using the sinc sampling kernel $\varphi[n]$ we will use the derivative sinc sampling

[2] $n_+ = n$, if $n \geq 0$, and 0 else.

kernel defined by $\psi[n] = (\underbrace{d * d * \cdots * d}_{R+1} * \varphi)[n]$ which has at least $R+1$ zeros at the origin $z = 1$. Then the DTFS of $\psi[n]$ is

$$\Psi[m] \quad = \quad (1 - W_N^m)^{R+1}\,\Phi[m], \quad m = 0, \ldots, N-1 \qquad (2.28)$$

where $\Phi[m]$ is the $Rect_{[-K(R+1),K(R+1)]}$ function. This brings us to the following theorem.

Theorem 2.1 *Consider a discrete-time periodic piecewise polynomial signal of period N with K pieces of degree R and with zero mean.* [3] *Let M be an integer and a divisor of N such that $N/M \geq (2K(R+1)+1)$. Take a sampling kernel $\psi[n]$ with DTFS coefficients defined in (2.28). Then we can recover the signal from the $N/M \in \mathbb{N}$ samples*

$$y_s[l] \quad = \quad < x[n], \psi[n - lM] > \quad l = 0, \ldots, N/M - 1. \qquad (2.29)$$

Proof: First we show that $2K(R+1)$ DTFS coefficients of the signal, $X[m], m \in [-K(R+1), K(R+1)]$ are sufficient to determine the piecewise polynomial signal, $x[n]$. Then we show that the N/M samples $y_s[l]$ are sufficient to determine the $(2K(R+1)+1)$ values $X[m]$.

1. If we have the DTFS coefficients $X[m], m \in [-K(R+1), K(R+1)]$ then from (2.27) we have the DTFS coefficients of the $(R+1)$th discrete-time differentiated signal, $X^{(R+1)}[m]$. From Prop. 2.1 these are sufficient to reconstruct the stream of $K(R+1)$ Diracs. Thus, the signal is recovered by applying $R+1$ times the inverse discrete-time difference operator, $d^{-1}[n]$, on the stream of Diracs, that is,

$$x[n] = \big(\underbrace{d^{-1} * d^{-1} * \cdots * d^{-1}}_{R+1} * x^{(R+1)}\big)[n].$$

2. Similar to the second part in the proof of Prop. 2.1 we expand the inner product between the piecewise polynomial signal and the differentiated

[3] We consider zero mean signals since $D \circ D^{-1}$ is a projector on the space of signals having zero mean.

sinc sampling kernel:

$$
\begin{aligned}
y_s[l] &= < x[n], \psi[n-lM] >, \quad l = 0, \ldots, N/M-1 && (2.30)\\
&= \sum_{n=0}^{N-1} x[n]\, \psi[n-lM] && (2.31)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, W_N^{-m(n-lM)} && (2.32)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, W_N^{-mn}\, W_N^{mlM} && (2.33)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, W_{N/M}^{ml} \underbrace{\sum_{n=0}^{N-1} x[n]\, W_N^{-nm}}_{X[-m]} && (2.34)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, X[-m]\, W_{N/M}^{ml}. && (2.35)
\end{aligned}
$$

Taking the DTFS of the sample values $y_s[l]$ we obtain

$$
\begin{aligned}
Y_s[k] &= \sum_{l=0}^{N/M-1} y_s[l]\, W_{N/M}^{lk}, \quad k = 0, \ldots, N/M-1 && (2.36)\\
&= \frac{1}{N} \sum_{l=0}^{N/M-1} \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, X[-m]\, W_{N/M}^{ml} W_{N/M}^{lk} && (2.37)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1-W_N^m)^{R+1}\, X[-m] \underbrace{\sum_{l=0}^{N/M-1} W_{N/M}^{l(k+m)}}_{=\begin{cases} N/M & \text{if } k+m=0 \\ 0 & \text{otherwise}\end{cases}} && (2.38)\\
&= \frac{1}{M} (1-W_N^{-k})^{R+1}\, X[k], \quad k = 0, \ldots, \min\{N/M, K(R+1)\} && (2.39)\\
\Rightarrow X[k] &= \begin{cases} M\left[(1-W_N^{-k})^{R+1}\right]^{-1} Y_s[k] & \text{for } k = 1, \ldots, K(R+1) \\ 0 & \text{for } k = 0 \end{cases} && (2.40)
\end{aligned}
$$

Since $N/M \geq (2K(R+1)+1)$ we have a sufficient representation for the spectral values of the signal. This completes the proof.

■

Figure 2.5 illustrates the reconstruction of a discrete-time periodic piecewise linear signal of period $N = 1024$ with $K = 6$ pieces.

2.2 Continuous-time periodic signals

We derive now the equivalent results but for continuous-time periodic signals, again building up from a stream of Diracs to piecewise polynomials. We will put in evidence the common points.

2.2.1 Stream of Diracs

Consider a continuous-time periodic signal $x(t)$ of period τ containing K weighted Diracs at locations $\{t_k\}_{k=0}^{K-1}$ with $t_k \in [0, \tau)$, or

$$\begin{aligned} x(t) &= \sum_{n\in\mathbb{N}} c_n\, \delta(t - t_n) \\ &= \sum_{n\in\mathbb{N}} \sum_{k=0}^{K-1} c_k\, \delta(t - (t_k + n\tau)) \end{aligned} \tag{2.41}$$

since $t_{n+K} = t_n + \tau$ and $c_{n+K} = c_n$ for all $n \in \mathbb{N}$.
The continuous-time Fourier series (CTFS) coefficients of $x(t)$ are defined by

$$\begin{aligned} X[m] &= \frac{1}{\tau}\int_0^{\tau} x(t)\, e^{-i2\pi tm/\tau}\, dt, \quad m \in \mathbb{Z} \\ &= \frac{1}{\tau}\sum_{k=0}^{K-1} c_k\, e^{-i2\pi t_k m/\tau}. \end{aligned} \tag{2.42}$$

If the signal $x(t)$ is convolved with a sinc filter of bandwidth $[-K, K]$ then we have a lowpass approximation $y(t)$ given by

$$y(t) = \sum_{m=-K}^{K} X[m]\, e^{i2\pi mt/\tau}. \tag{2.43}$$

Suppose the lowpass approximation $y(t)$ is sampled at multiples of T, we obtain $\tau/T \in \mathbb{N}$ samples defined by

$$y_s[l] = y(lT) = \sum_{m=-K}^{K} X[m]\, e^{i2\pi mlT/\tau}, \quad l = 0, \dots, \tau/T - 1. \tag{2.44}$$

Similar to the discrete-time case as long as the number of samples is larger than the number of values in the spectral support of the lowpass signal, that is, $\frac{\tau}{T} \geq 2K + 1$, (2.44) can be used to recover $2K + 1$ values of $X[m]$. Thus we can state:

Proposition 2.2 *Consider a continuous-time periodic stream of K weighted Diracs with period τ and a continuous-time periodic sinc sampling kernel $\varphi(t)$ with bandwidth $[-K, K]$. Taking a sampling period T such that $\tau/T \in \mathbb{N}$ and $\tau/T \geq 2K + 1$. Then the samples defined by*

$$y_s[l] = \langle x(t), \varphi(t - lT) \rangle, \quad l = 0, \dots, \tau/T - 1 \tag{2.45}$$

are a sufficient representation of $x(t)$.

Proof: Similar to the discrete-time case first we show that $2K+1$ CTFS coefficients $X[m]$ are sufficient to find the locations and the weights of the Diracs. From (2.42) we have that the CTFS coefficients $X[m]$ are linear combinations of complex exponentials. Thus to find the locations t_k we need to find the annihilating filter $\mathbf{H} = (1, H[1], H[2], \ldots, H[K])$ such that

$$\mathbf{H} *_c \mathbf{X} \;=\; 0. \tag{2.46}$$

This is the same Toeplitz system as in (2.10) considered in Sec. 2.1.1 and therefore a solution exists. Factoring the $z-$transform of $\mathbf{H}$, or $H(z) = \sum_{k=0}^{K} H[k]\, z^{-k}$, into

$$H(z) \;=\; \prod_{k=0}^{K-1} (1 - z^{-1} u_k), \tag{2.47}$$

we then find the K locations $\{t_0, t_1, \ldots, t_{K-1}\}$ from the zeros of $H(z)$, that is, from

$$u_k \;=\; e^{-i2\pi t_k/\tau}. \tag{2.48}$$

Given the locations $\{t_k\}_{k=0}^{K-1}$ and K values $X[m], m = 0, \ldots, K-1$, we find the weights $\{c_k\}_{k=0}^{K-1}$ of the Diracs by solving the Vandermonde system in (2.42). Since the locations t_k are distinct, $t_k \neq t_l, \forall k \neq l$, the Vandermonde system admits a solution.
The second part of the proof consists in showing that the τ/T samples $y_s[l]$ are sufficient to determine the CTFS coefficients $X[m], m \in [-K, K]$. We substitute the continuous-time periodic sinc function $\varphi(t)$ with bandwidth $[-K, K]$ defined by

$$\varphi(t) = \sum_{m=-K}^{K} e^{i2\pi m t}. \tag{2.49}$$

in (2.45) and we obtain

$$\begin{aligned}
y_s[l] &= \; < x(t), \varphi(t - lT) >, \quad l = 0, \ldots, \tau/T - 1 && (2.50)\\
&= \int_0^{\tau} x(t) \sum_{m=-K}^{K} e^{i2\pi m(t-lT)/\tau}\, dt && (2.51)\\
&= \sum_{m=-K}^{K} e^{-i2\pi m l/(\tau/T)} \underbrace{\int_0^{\tau} x(t) e^{i2\pi m t/\tau}\, dt}_{\tau X[-m]} && (2.52)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, e^{-i2\pi m l T/\tau}. && (2.53)
\end{aligned}$$

Note that $y_s[l]$ is periodic with period τ/T, thus the DTFS coefficients are $Y_s[k] = TX[k], k = 0, \ldots, \tau/T - 1$. Since $\tau/T \geq 2K+1$, we have a sufficient number of samples that determine the CTFS $X[m], m \in [-K, K]$. ■

2.2.2 Piecewise polynomials of degree R

Here we consider continuous-time periodic piecewise polynomial signal of period τ, containing K pieces of maximum degree R and $R-1$ continuous derivatives, C^{R-1},

$$x(t) = \frac{1}{R!}\sum_{k=0}^{K-1} c_k\,(t-t_k)_+^R, \quad t \in [0,\tau]. \tag{2.54}$$

We differentiate the signal $R+1$ times and we obtain a continuous-time stream of K weighted Diracs, $x^{(R+1)}(t)$. The CTFS of the derivative operator is defined by $D[m] = i2\pi m, m \in \mathbb{Z}$ and therefore the CTFS coefficients of the differentiated signal $x^{(R+1)}(t)$ are equal to

$$X^{(R+1)}[m] = (i2\pi m)^{R+1}\,X[m], \quad m \in \mathbb{Z}. \tag{2.55}$$

From Proposition 2.2 we can recover the continuous-time periodic stream of K Diracs from the CTFS coefficients, $X^{(R+1)}[m], m \in [-K,K]$. Therefore we can sample the signal with the differentiated sinc sampling kernel whose CTFS coefficients are defined by

$$\Psi[m] = (i2\pi m)^{R+1}\,\Phi[m], \quad m \in \mathbb{Z} \tag{2.56}$$

where $\Phi[m] = Rect_{[-K,K]}$ is the CTFS of the continuous-time periodized sinc sampling kernel.

Theorem 2.2 *Consider a continuous-time periodic piecewise polynomial signal $x(t)$ with period τ, containing K pieces of maximum degree R, belonging to C^{R-1} and having zero mean. Consider a sampling kernel $\psi(t)$ with its CTFS coefficients defined in (2.56). Let $\tau/T \in \mathbb{N}$ and $\tau/T \geq 2K+1$. Then $x(t)$ can be uniquely recovered from the τ/T samples*

$$y_s[l] = < x(t), \psi(t-lT) >, \quad l = 0,\ldots,\tau/T-1. \tag{2.57}$$

Proof: Similar to the proof of Theorem. 2.1, we first show that CTFS coefficients $X[m], m \in [-K,K]$ are sufficient to determine the piecewise polynomial signal, $x(t)$. Then we show that the τ/T samples $y_s[l]$ are sufficient to determine the values $X[m], m \in [-K,K]$.

1. If we have the CTFS coefficients $X[m], m \in [-K,K]$ then from (2.55) we have the CTFS coefficients of the $(R+1)$th differentiated signal, $X^{(R+1)}[m]$. From Prop. 2.2 these are sufficient to reconstruct the stream of K Diracs. Thus, the signal is recovered by integrating $R+1$ times the stream of Diracs, that is,

$$x(t) = \underbrace{\int\int\cdots\int}_{R+1} x^{(R+1)}(t)\,dt\,dt\cdots dt$$

or in frequency domain from (2.55)

$$\begin{aligned} X[m] &= (D^{-1}[m])^{R+1}\,X^{(R+1)}[m], \quad m \in \mathbb{Z}/\{0\} &(2.58)\\ &= (i2\pi m)^{-(R+1)}\,X^{(R+1)}[m], \quad m \in \mathbb{Z}/\{0\} &(2.59)\end{aligned}$$

with $D^{-1}[m] = 0$ for $m = 0$ and thus

$$x(t) = \sum_{m \in \mathbb{Z}} X[m]\, e^{i2\pi m t/\tau}.$$

2. Similar to the second part in the proof of Prop. 2.2 we expand the inner product between the piecewise polynomial signal and the differentiated sinc sampling kernel defined by $\psi(t) = \sum_{m=-K}^{K} (i2\pi m)^{R+1} e^{i2\pi m t/\tau}$. That is, the sample values are

$$\begin{aligned}
y_s[l] &= \langle x(t), \psi(t - lT) \rangle, \quad l = 0, \ldots, \tau/T - 1 && (2.60)\\
&= \int_0^\tau x(t) \sum_{m=-K}^{K} (i2\pi m)^{R+1} e^{i2\pi m(t-lT)/\tau}\, dt && (2.61)\\
&= \sum_{m=-K}^{K} (i2\pi m)^{R+1}\, e^{-i2\pi m l T/\tau} \underbrace{\int_0^\tau x(t)\, e^{i2\pi m t/\tau}\, dt}_{\tau X[-m]} && (2.62)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1}\, e^{-i2\pi m l/(\tau/T)}. && (2.63)
\end{aligned}$$

Since $y_s[l]$ is periodic with period τ/T, the DTFS coefficients of $y_s[l]$ are given by

$$\begin{aligned}
Y_s[k] &= \sum_{l=0}^{\tau/T-1} y_s[l]\, e^{-i2\pi k l/(\tau/T)} && (2.64)\\
&= \tau \sum_{l=0}^{\tau/T-1} \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1}\, e^{-i2\pi m l/(\tau/T)}\, e^{-i2\pi k l/(\tau/T)} && (2.65)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1} \underbrace{\sum_{l=0}^{\tau/T-1} e^{-i2\pi(k+m)l/(\tau/T)}}_{=\begin{cases} \tau/T & \text{if } k+m=0 \\ 0 & \text{otherwise} \end{cases}} && (2.66)\\
&= \frac{\tau^2}{T} (-i2\pi m)^{R+1}\, X[k] && (2.67)
\end{aligned}$$

Therefore the CTFS coefficients of the signal are obtained by the DTFS coefficients of the samples values $Y_s[m], m = 0, \ldots, \tau/T - 1$ and are defined by

$$X[m] = \begin{cases} T\, Y_s[m]/(\tau^2\, (-i2\pi m)^{R+1}) & \text{for } m = 1, \ldots, \tau/T - 1 \\ 0 & \text{for } m = 0 \end{cases}. \quad (2.68)$$

Since $\tau/T \geq 2K + 1$ the sample values are a sufficient representation of the spectral values of the signal. This completes the proof.

■

Note that removing the restriction $x(t) \in C^{R-1}$ leads to the same result as in Theorem. 2.1.

2.3 Applications

The applications we consider involve the discrete-time periodic stream of Diracs and piecewise polynomial signals. It is well known that a bandlimited signal can be perfectly recovered from its samples by sampling it at twice the maximum frequency. What if the bandlimited signal has a jump or a discontinuity then the signal is no longer bandlimited and the usual method is not valid. These are what we call piecewise bandlimited signals. Another type of non-bandlimited signal which we may come across in nature is a signal which is obtained from a system with a certain frequency response. The output of the system is a filtered signal. We will look at filtered stream of Diracs and filtered piecewise polynomials.

2.3.1 Piecewise bandlimited signals

A discrete-time periodic piecewise bandlimited signal is the sum of a bandlimited signal with a stream of Diracs in the simplest case or with a piecewise polynomial signal. An example is illustrated in Figure 2.6(e) and is obviously not bandlimited from Figure 2.6(f). Formally, we have the following

Definition 2.1 *Piecewise bandlimited signals.*
Let $\mathbf{x}_{BL}$ be a discrete-time periodic $L-$bandlimited signal of period N with corresponding DTFS coefficients $\mathbf{X}_{BL}$ such that $X_{BL}[m] = 0 \quad \forall m \notin [-L, L]$. Let $\mathbf{x}_{PP}$ be a zero mean discrete-time piecewise polynomial signal of period N with K pieces and with each piece of maximum degree R. Then a piecewise bandlimited signal $\mathbf{x}$ is defined by

$$\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP} \tag{2.69}$$

with corresponding DTFS coefficients $\mathbf{X}$ defined by

$$X[m] = \begin{cases} X_{BL}[m] + X_{PP}[m] & \text{if } m \in [-L, L] \\ X_{PP}[m] & \text{if } m \notin [-L, L] \end{cases} . \tag{2.70}$$

First consider a stream of K weighted Diracs, $\mathbf{x}_{PP}$. From Section 2.1.1, we can recover the K weighted Diracs from $2K$ contiguous frequency values $\mathbf{X}_{PP}$. Since the DTFS coefficients of the bandlimited signal, $\mathbf{X}_{BL}$, are equal to zero outside of the band $[-L, L]$, we have that the DTFS coefficients of the signal outside of the band $[-L, L]$ are exactly equal to the DTFS coefficients of the piecewise polynomial, that is, $\mathbf{X}[m] = \mathbf{X}_{PP}[m], \quad \forall |m| > L$. Therefore it is sufficient to take the $2K$ DTFS coefficients of the outside of the band $[-L, L]$, for instance in $[L+1, L+2K]$. Suppose we have the DTFS of the signal $X[m]$, with $m \in [-(L+2K), L+2K]$ then the DTFS of the bandlimited signal are obtained by subtracting $X_{PP}[m]$ from $X[m]$ for $m \in [-L, L]$.
Recall that the piecewise polynomial has $2K(R+1)$ degrees of freedom and the bandlimited signal has $2L+1$. It follows that we can sample the signal using a discrete-time periodized differentiated sinc sampling kernel bandlimited to $2K(R+1) + L$.

Corollary 2.1 *Consider a piecewise bandlimited signal* $\mathbf{x}$ *as defined in Definition. 2.1. Let* $\psi[n]$ *be the* $(R+1)$*th differentiated sinc sampling kernel with DTFS*

$$\Psi[m] = (D[m])^{R+1}\, Rect_{[-(2K(R+1)+L),2K(R+1)+L]}. \tag{2.71}$$

Let M *be an integer divisor of* N*, and let* $N/M \geq 2(2K(R+1)+L)$ *then the samples*

$$y_s[l] = < x[n], \psi[n-lM] >, \quad l = 0, \ldots, N/M - 1 \tag{2.72}$$

are a sufficient representation of $\mathbf{x}$.

Proof: The proof is exactly the same as in Theorem 2.1 until equation (2.39)

$$Y_s[k] = \frac{1}{M}(1-W_N^{-k})^{R+1}\, X[k], k = 0, \ldots, 2K(R+1)+L \tag{2.73}$$

$$= \begin{cases} \frac{1}{M}(1-W_N^{-k})^{R+1}\,(X_{BL}[k] + X_{PP}[k]) & \text{if } k = 0, \ldots, L \\ \frac{1}{M}(1-W_N^{-k})^{R+1}\, X_{PP}[k] & \text{if } k = L+1, \ldots, 2K(R+1)+L \end{cases} \tag{2.74}$$

Therefore $2K(R+1)$ values of

$$X_{PP}[k] = \frac{M}{(1-W_N^{-k})^{R+1}} Y_s[k], \quad k \in [L+1, 2K(R+1)+L] \tag{2.75}$$

are sufficient to recover the piecewise polynomial $\mathbf{x}_{PP}$. From these we can recover the L spectral components of the bandlimited signal since

$$X_{BL}[k] = \frac{1}{(1-W_N^{-k})^{R+1}}(Y_s[k] - X_{PP}[k]), \quad k = 0, \ldots, L. \tag{2.76}$$

This gives us the the bandlimited signal $\mathbf{x}_{BL}$ and thus the piecewise bandlimited signal as defined in Definition. 2.1 is recovered $\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP}$. ∎

Figure 2.7 the illustrates the reconstruction of a bandlimited plus a piecewise constant signal using the following reconstruction scheme:

Algorithm 2.1 *Reconstruction of piecewise bandlimited signals.*
Require: $N, M, N/M \geq 2(2K(R+1)+L)+1$;
Calculate the samples $y_s[l] = < x[n], \psi[n-lM] >, l = 0, \ldots, N/M-1$;
Calculate the DTFS $X[m], m \in [-(2K(R+1)+L), (2K(R+1)+L)]$ *from the DTFS of samples* $y_s[l] \longrightarrow X_{PP}[m] = X[m], m \in [L+1, (2K(R+1)+L)]$;
Solve $\mathbf{h} * X_{PP}[m] = 0, m \in [L+1, (2K(R+1)+L)] \longrightarrow \mathbf{x}_{PP}$;
Calculate $X_{BL}[m] = X[m] - X_{PP}[m], \quad m \in [-L, L] \longrightarrow \mathbf{x}_{BL}$;
The reconstruction is $\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP}$.

2.3.2 Filtered piecewise polynomials

Another application of sampling piecewise polynomial signals consists in sampling their filtered output. Figure 2.8 illustrates that a filtered stream of Diracs is not bandlimited. These signals are formally defined in the following

Definition 2.2 *Filtered piecewise polynomials.*
Let $\mathbf{x}_{PP}$ be a zero mean discrete-time periodic piecewise polynomial signal of period N with K pieces of maximum degree R. Let $\mathbf{g}$ be a filter with DTFS $\mathbf{G}$. Then a filtered piecewise polynomial $\mathbf{x}$ is defined by

$$\mathbf{x} = \mathbf{g} * \mathbf{x}_{PP} \tag{2.77}$$

and the corresponding DTFS coefficients $\mathbf{X}$ are defined by

$$X[m] = G[m] \cdot X_{PP}[m], \quad m = 0, \ldots, N-1. \tag{2.78}$$

Suppose $\mathbf{x}_{PP}$ is a stream of K Diracs. If the filter has $2K$ contiguous nonzero frequency values $G[m]$ then $2K$ frequency values of the signal $X[m]$ will be enough to determine $2K$ frequency values of the stream of Diracs, since $X_{PP}[m] = X[m]/G[m]$, and from Prop. 2.1 these are sufficient to recover the stream of Diracs .

Corollary 2.2 *Consider a filtered piecewise polynomial signal $\mathbf{x}$ as defined in Definition. 2.2 with $G[m] \neq 0, m \in [-K(R+1), K(R+1)]$. Consider an $(R+1)$ differentiated sinc sampling kernel $\psi[n]$ with DTFS*

$$\Psi[m] = (D[m])^{R+1}\, Rect_{[-K(R+1),K(R+1)]}. \tag{2.79}$$

Let M be an integer divisor of N such that $N/M \geq 2K(R+1)+1$. Then the filtered piecewise polynomial signal can be recovered from the N/M samples

$$y_s[l] = < x[n], \psi[n-lM] >, \quad l = 0, \ldots, N/M - 1. \tag{2.80}$$

Proof: Similar to the proof of piecewise bandlimited signals, we have that the DTFS coefficients of the samples $y_s[l]$ are equal to

$$\begin{aligned} Y_s[k] &= \frac{1}{M}(1 - W_N^{-k})^{R+1}\, X[k], \quad k \in [-K(R+1), K(R+1)] && (2.81)\\ &= \frac{1}{M}(1 - W_N^{-k})^{R+1}\, \left(G[k]\, X_{PP}[k]\right). && (2.82) \end{aligned}$$

Since $G[k] \neq 0$ for $k \in [-K(R+1), K(R+1)]$ we have $2K(R+1)$ values of the DTFS of the piecewise polynomial

$$X_{PP}[k] = \frac{M}{(1 - W_N^{-k})^{R+1}\, G[k]}\, Y_s[k], \quad k \in [-K(R+1), K(R+1)] \tag{2.83}$$

which are sufficient to recover $\mathbf{x}_{PP}$ and which leads to the filtered signal by Definition. 2.2. ∎

The reconstruction scheme is described in the following algorithm and an example of the reconstruction is illustrated in Figure 2.9.

Algorithm 2.2 ***Require:*** $N, M, N/M \geq 2K(R+1)+1$;
Calculate the samples $y_s[l] = < x[n], \psi[n-lM] >, l = 0, \ldots, N/M-1$;
Calculate $\mathbf{Y}_s = \mathbf{DFT}_{N/M} \cdot \mathbf{y}_s \longrightarrow X[m], m \in [-K(R+1), K(R+1)]$;
Calculate $X_{PP}[m] = X[m]/G[m], m \in [-K(R+1), K(R+1)]$;
Solve $\mathbf{h} * X_{PP}[m] = 0, m \in [-K(R+1), K(R+1)] \longrightarrow \mathbf{x}_{PP}$;
The reconstruction is $\mathbf{x} = \mathbf{g} * \mathbf{x}_{PP}$.

We have seen that the crux of the proof relies on the fact that the filter is known and is invertible over the number of degrees of freedom of the problem. What if the filter has a finite rate of innovation but is unknown? This is more complex and remains to be investigated.

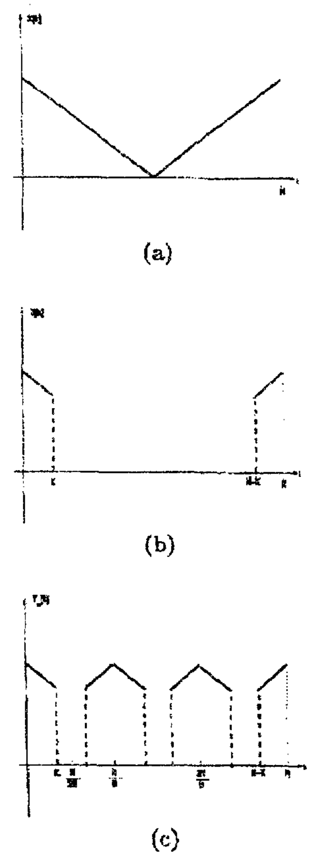


Figure 2.1: (a)DTFS of stream of Diracs, $X[k], k \in [0, N]$; (b) DTFS of lowpass approximation $Y[k] = X[k], k \in [-K, K], 0$ otherwise; (c) DTFS of lowpass approximation subsampled by $M = 3$, $Y_s[k] = 1/MX[k], k \in [-K, K]$.

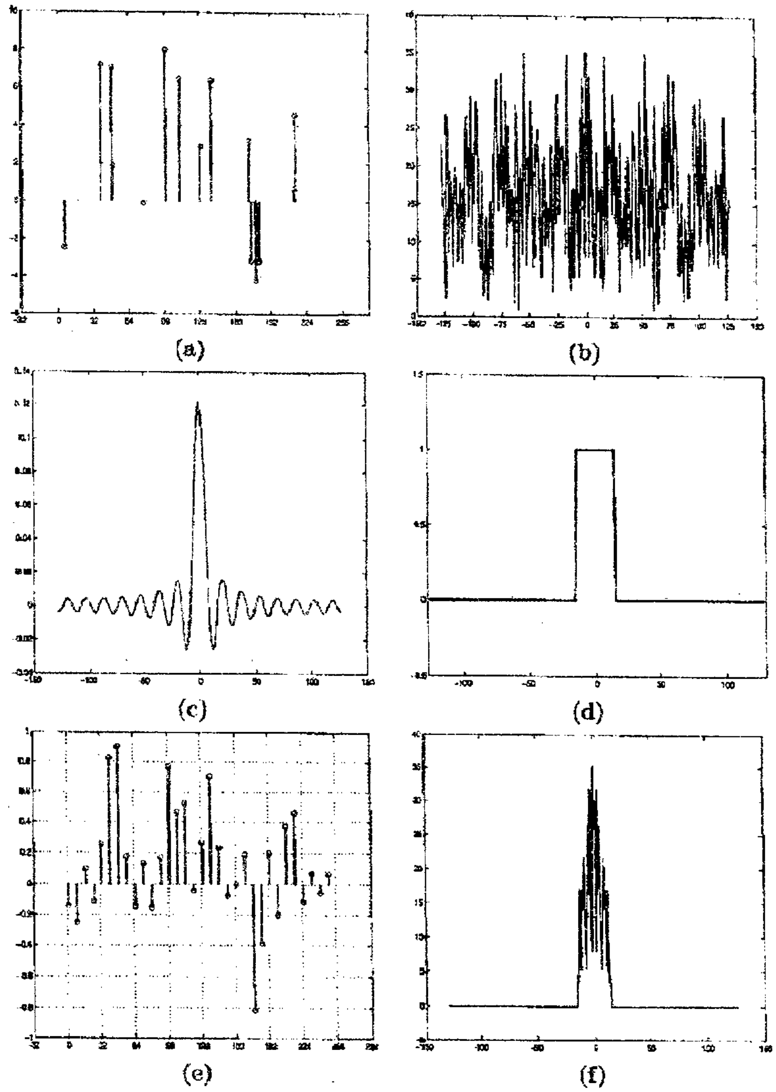


Figure 2.2: (a) Periodic discrete-time signal with $K = 15$ weighted Diracs with period $N = 256$;(b) DTFS $X[m]$; (c) Discrete-time periodized sinc sampling kernel, $\varphi[n]$; (d) DTFS $Rect_{[-K,K]}$, $K = 15$(e) Sample values $y_s[l] =< x[n], \varphi[n - lM] >$,$l = 0, \dots, 31$ with $M = 8$; (f) DTFS $\mathbf{Y}_s$.

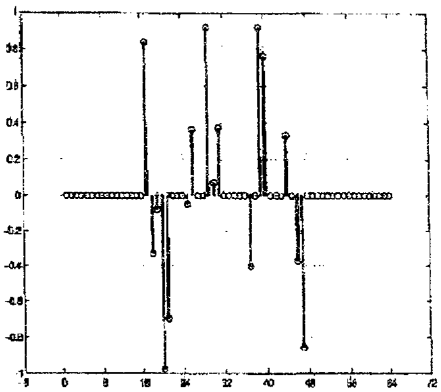


Figure 2.3: Stream of $K = 16$ bunched Diracs with period $N = 64$.

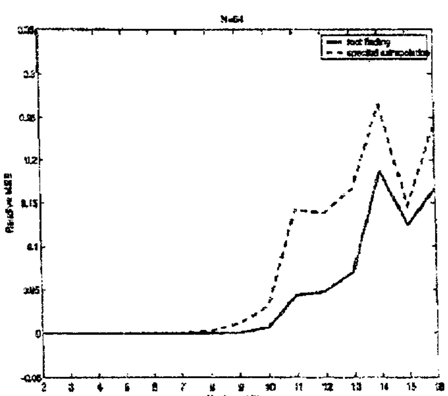


Figure 2.4: Comparison between the root finding method and the spectral extrapolation method on a signal of length $N = 64$, K varying between 2 and 16 on interval $2K$, 100 simulations.

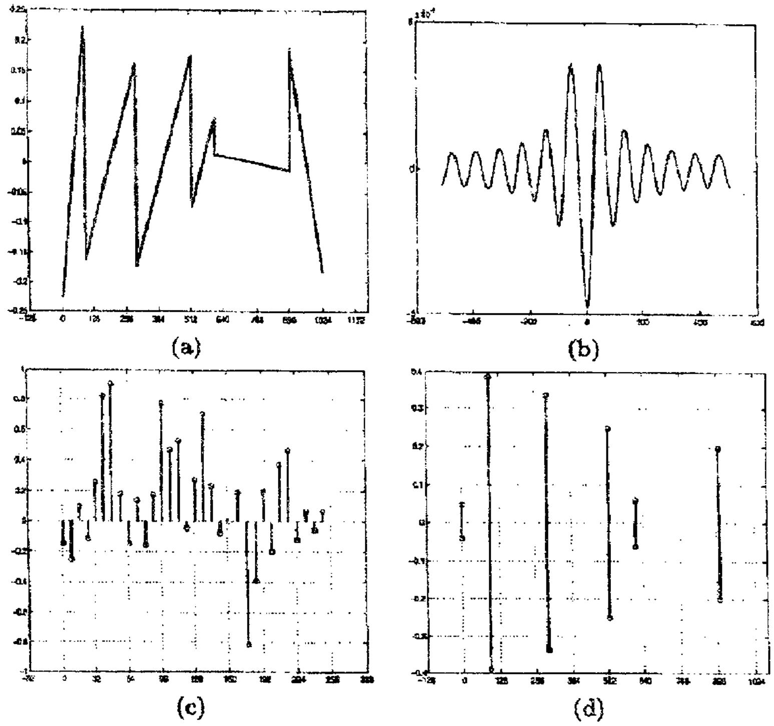


Figure 2.5: (a) Discrete-time periodic piecewise linear ($R = 1$) signal of period $N = 1024$ with $K = 6$ pieces; (b) Differentiated sinc sampling kernel,$\psi[n] = d[n]*d[n]*\varphi[n]$ with DTFS $D[m]\cdot Rect_{[-K(R+1),K(R+1)]}$ (c) Sample values $y_s[l] =< x[n], \psi[n-lM] >, l = 0,\ldots,31$ with $M = 32$; (d) Stream of $K(R+1) = 12$ Diracs obtained from $X[m], m \in [-K(R+1), K(R+1)]$.

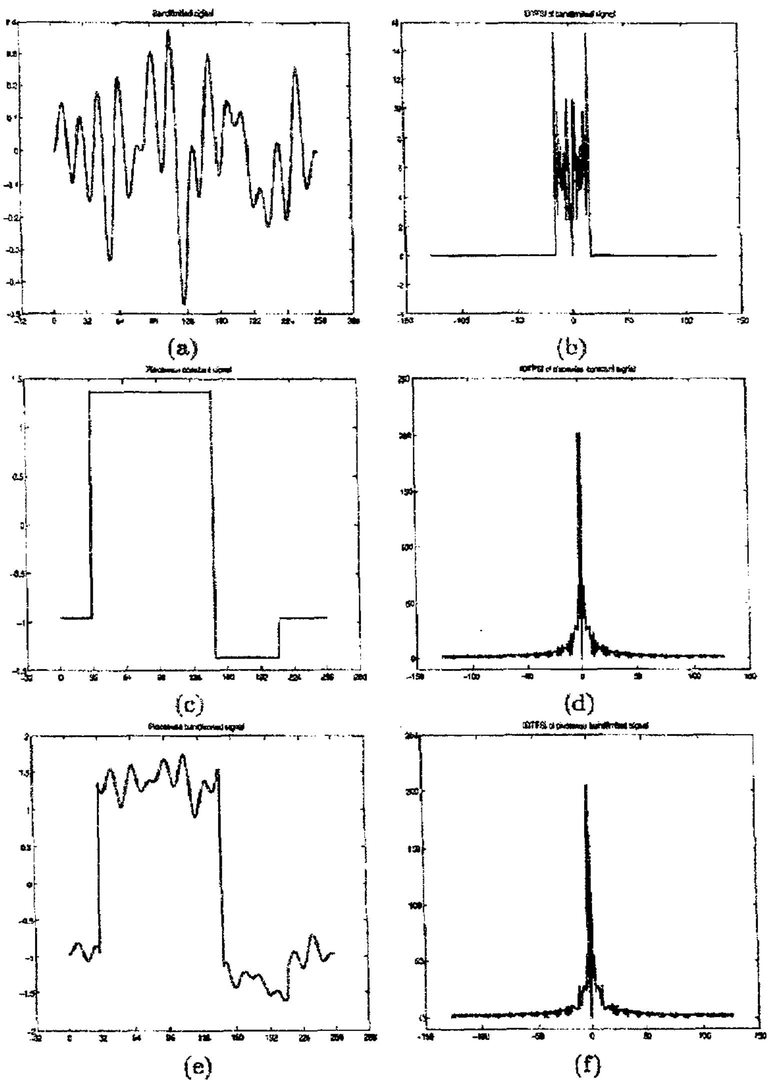


Figure 2.6: (a) Bandlimited signal of length $N = 256$; (b) DTFS of Bandlimited signal, $L = 15$ (c) Piecewise constant signal with $K = 3$ pieces; (d) DTFS of piecewise constant signal; (e) Bandlimited piecewise constant signal; (f) DTFS of bandlimited piecewise constant signal.

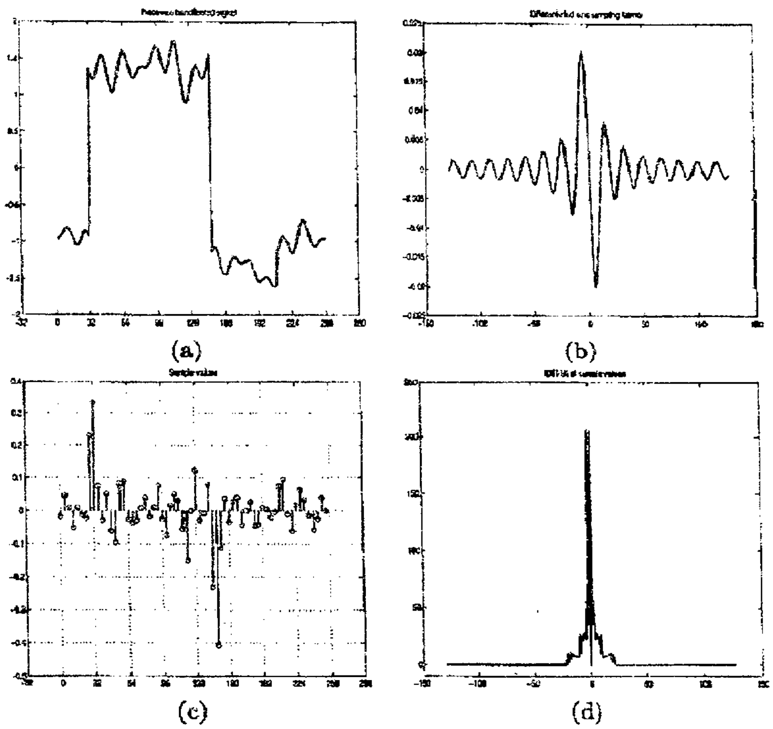


Figure 2.7: (a) Bandlimited piecewise constant signal,$x[n]$, with $K = 3$, $L = 15$,$R = 0$,$N = 256$; (b) Differentiated sinc sampling kernel, $\psi[n] = d[n] * \varphi[n]$, bandlimited to $2K(R+1)+1+L = 22$ (c) Sample values $y_s[l] = < x[n], \psi[n-lM] >$,$l = 0, \ldots, N/M - 1$, $M = 4$; (d) $|DTFS|$ of sample values; Reconstruction error is 10^{-13}.

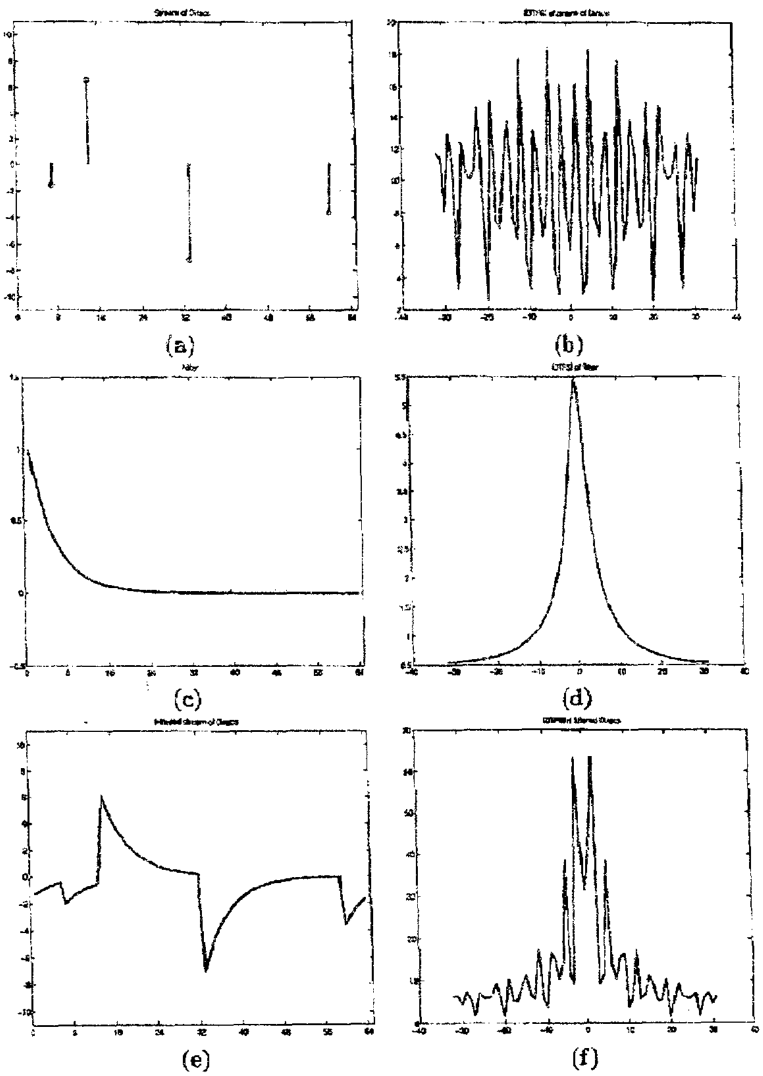


Figure 2.8: (a) Stream of $K = 4$ Diracs with period $N = 64$; (b) $|DTFS|$ of stream of Diracs (c) Known filter $g[n] = \alpha^n, n = 0, \ldots, N-1$, $\alpha = 0.4$; (d) $|DTFS|$ of filter; (e) Filtered stream of Diracs; (f) $|DTFS|$ of filtered stream of Diracs.

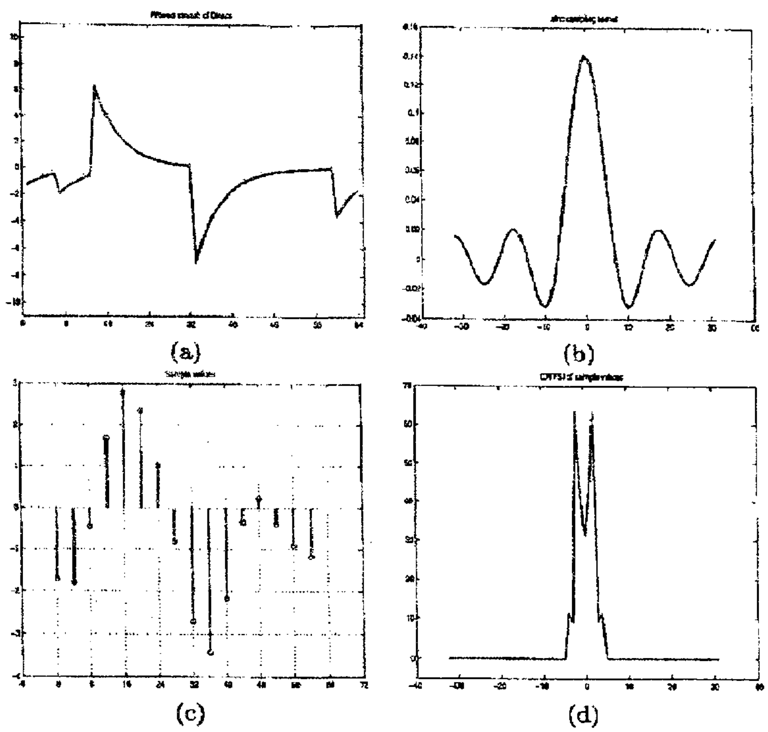


Figure 2.9: (a) Filtered stream of Diracs, $x[n]$, $N = 64$; (b) Sinc sampling kernel $\varphi[n]$ bandlimited to $K = 4$ (c) Sample values $y_s[l] =< x[n], \varphi[n - lM] >, l = 0, \dots, 15$, $M = 4$; (d) $|DTFS|$ of sample values; Reconstruction error is 10^{-13}.

2.4 Summary

- We derived sampling theorems for periodic signals in particular streams of weighted Diracs and piecewise polynomials. These signals have a finite rate of innovation ρ which is equal to the number of degrees of freedom per period.
- The samples are obtained by taking the inner product of the signal with a shifted version of the periodized sinc kernel or differentiated sinc kernels. The bandwidth of these kernels must be greater or equal to the degrees of freedom of the signal.
- The discrete-time periodic signals are perfectly recovered when the sampling rate $1/M$ is greater or equal to the rate of innovation $\rho = 2K/N$ in the case of streams of weighted Diracs or $\rho = 2K(R+1)/N$ in the case of a piecewise polynomial signal with K pieces and maximum degree R.
- The continuous-time periodic streams of Diracs and piecewise polynomial signals are perfectly recovered when the sampling rate $1/T$ is greater or equal to the rate of innovation $\rho = 2K/\tau$ since we assumed that the piecewise polynomial signal belonged to C^{R-1}.
- The sampling and reconstruction scheme is illustrated in Figure 2.10.

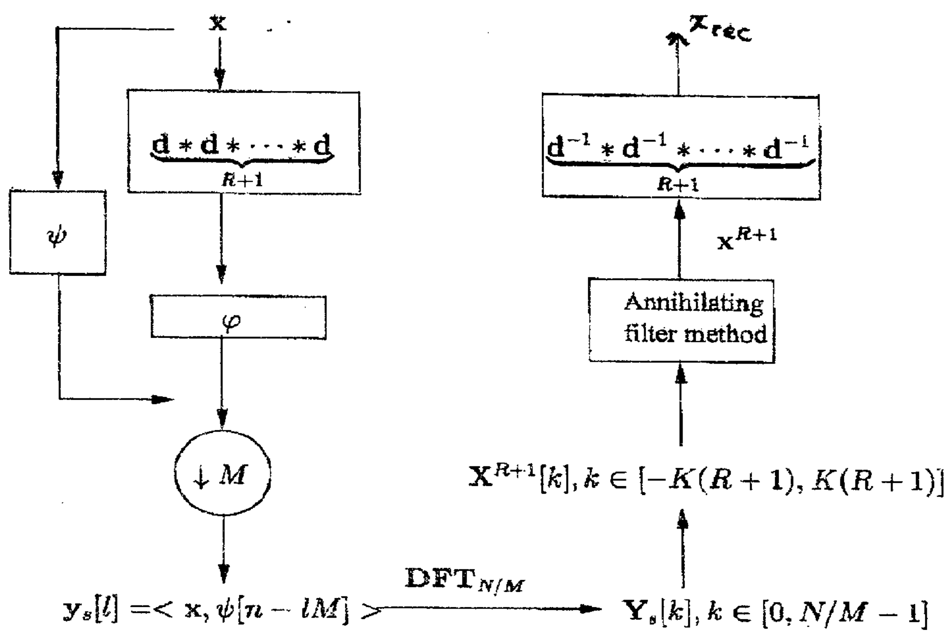


Figure 2.10: Sampling and reconstruction scheme for discrete-time piecewise polynomial signals with K pieces of maximum degree R; N/M is the number of samples; $2K(R+1)+1$ is the bandwidth of the sampling kernel.

Chapter 3

Sampling signals with finite rate and finite local rate of innovation

In this chapter[1] we go beyond periodicity in terms of the signals and the sampling kernels. Section 3.1 investigates sampling finite length signals with a finite rate of innovation using sampling kernels with infinite support. The signals in question are streams of weighted Diracs sampled with the sinc and the Gaussian kernel. These types of kernels are appealing to mathematicians. It will be shown that if the critical number of samples is taken then a sampling theorem can be derived. Section 3.2 considers the dual problem: Infinite length piecewise polynomial signals and compact support sampling kernels. A particular interest is given to bilevel signals with a finite *local* rate of innovation and spline sampling kernels. Given that the signals have a finite local rate of innovation, local reconstruction is possible and schemes are given in Section 3.2.3.

3.1 Finite length signals with finite rate of innovation

A finite length signal with finite rate of innovation ρ clearly has a finite number of degrees of freedom. The question of interest is: Given a sampling kernel with *infinite support*, is there a *finite set of samples* that uniquely specifies the signal? In the following sections we will sample signals with finite number of weighted Diracs with infinite support sampling kernels such as the sinc and Gaussian.

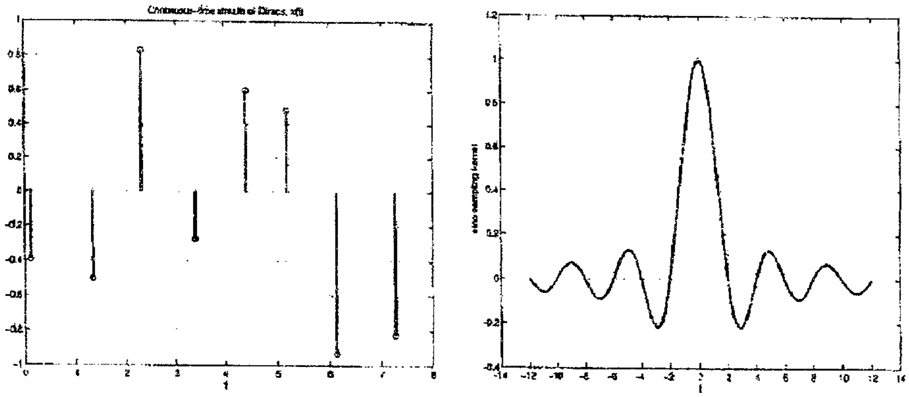
3.1.1 Sinc sampling kernel

Consider a continuous-time signal with a finite number of weighted Diracs

$$x(t) = \sum_{k=0}^{K-1} c_k \, \delta(t - t_k) \tag{3.1}$$

[1]This chapter includes research conducted jointly with Martin Vetterli and Thierry Blu [80, 78].

and an infinite length sinc sampling kernel, see Figure 3.1. The sample values



(a) (b)

Figure 3.1: (a) Example of a finite length continuous-time stream of $K = 8$ Diracs randomly spread on an interval $[0, \tau]$ with $\tau = 8$; (b) Sinc sampling kernel, $\text{sinc}(t/T), T = 2$.

are obtained by filtering the signal with a $\text{sinc}(t/T), t \in \mathbb{R}$, sampling kernel. This is equivalent to taking the inner product between the signal and a shifted version of the sinc

$$y_n = < x(t), \text{sinc}(t/T - n) > \tag{3.2}$$

where $\text{sinc}(t) = \sin(\pi t)/\pi t$. The question that arises is: How many of these samples do we need to recover the signal? The signal has $2K$ degrees of freedom, K from the weights and K from the locations of the Diracs and thus N samples, $N \geq 2K$, will be sufficient to recover the signal. Similar to the previous cases, the reconstruction method will require solving two systems of linear equations: one for the locations of the Diracs and the second for the weights of the Diracs. These systems admit solutions if the following conditions are satisfied:

[C1] $\text{Rank}(\mathbf{V}) < K+1$ where $v_{nk} = \Delta^K \left((-1)^n n^k y_n\right)$ and $\mathbf{V} \in \mathbb{R}^{(N-K)\times(K+1)}$;

[C2] $\text{Rank}(\mathbf{A}) = K$ where $a_{nk} = \frac{\sin(\pi t_k/T)}{\pi(t_k/T - n)}$ and $\mathbf{A} \in \mathbb{R}^{K\times K}$.

Theorem 3.1 *Given a finite stream of K weighted Diracs and a sinc sampling kernel $\text{sinc}(t/T)$. If conditions [C1] and [C2] are satisfied then N samples with $N \geq 2K$*

$$y_n \;=\; < x(t), \text{sinc}(t/T - n) > \tag{3.3}$$

are a sufficient representation of the signal.

Proof: Taking the inner products between the signal and shifted versions of the sinc sampling kernel yields a set of N samples

$$
\begin{aligned}
y_n &= \langle x(t), \mathrm{sinc}(t/T - n) \rangle, \quad n = 0, \ldots, N-1 && (3.4)\\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{K-1} c_k\, \delta(t - t_k)\, \mathrm{sinc}(t/T - n)\, dt && (3.5)\\
&= \sum_{k=0}^{K-1} c_k\, \mathrm{sinc}(t_k/T - n) && (3.6)\\
&= \sum_{k=0}^{K-1} \frac{c_k \sin(\pi t_k/T - \pi n)}{\pi(t_k/T - n)} && (3.7)\\
&= (-1)^n \sum_{k=0}^{K-1} \frac{c_k \sin(\pi t_k/T)}{\pi(t_k/T - n)} && (3.8)\\
\Leftrightarrow (-1)^n y_n &= \frac{1}{\pi} \sum_{k=0}^{K-1} c_k \sin(\pi t_k/T) \cdot \frac{1}{(t_k/T - n)} && (3.9)
\end{aligned}
$$

The denominator of the previous expression (3.9) can be rewritten as follows:

$$
\frac{1}{(t_k/T - n)} = \frac{\prod_{l \neq k} (t_l/T - n)}{\prod_{l=0}^{K-1} (t_l/T - n)} = \frac{P_k(n)}{P(n)} \qquad (3.10)
$$

where $P(u)$ is a polynomial of degree K with zeros at all values of t_k/T,

$$
P(u) = \prod_{l=0}^{K-1} (t_l/T - u) = \sum_{k=0}^{K} p_k\, u^k \qquad (3.11)
$$

and the $P_k(u)$ is a polynomial of degree $K-1$ and has zeros at all locations except at location t_k

$$
P_k(u) = \prod_{l \neq k} (t_l/T - u). \qquad (3.12)
$$

Therefore if the coefficients of the polynomial $P(u)$ are determined then the locations of the Diracs are simply the K roots of $P(u)$. We can now find an equivalent expression to (3.9) in terms of the interpolating polynomials:

$$
(-1)^n P(n)\, y_n = \frac{1}{\pi} \sum_{k=0}^{K-1} c_k \sin(\pi t_k/T)\, P_k(n). \qquad (3.13)
$$

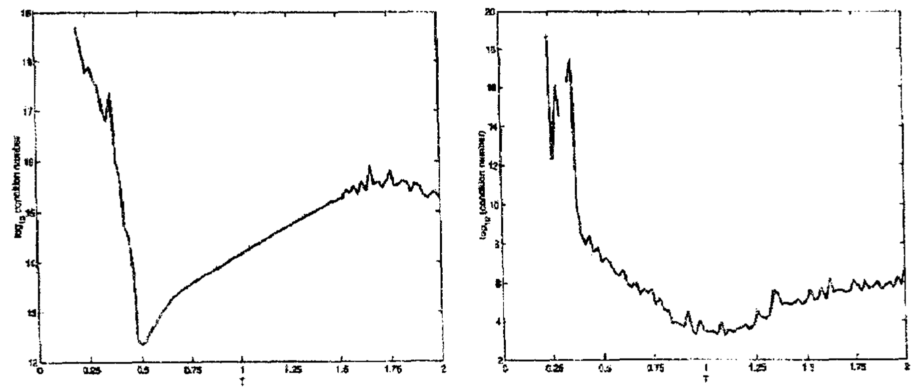
Note that the right-hand side of (3.13) is a polynomial of degree $K-1$ in the variable n, applying K finite differences makes the left-hand side vanish,[2] that is,

$$
\begin{aligned}
&\Delta^K\left((-1)^n P(n)\, y_n\right) = 0, \quad n = K, \ldots, N-1 && (3.14)\\
\Leftrightarrow\ & \textstyle\sum_{k=0}^{K} p_k \underbrace{\Delta^K\left((-1)^n n^k y_n\right)}_{v_{nk}} = 0 && (3.15)\\
\Leftrightarrow\ & \mathbf{V} \cdot \mathbf{p} = 0 && (3.16)
\end{aligned}
$$

[2] Note that the K finite difference operator plays the same role as the annihilating filter in the previous chapter.

where the matrix $\mathbf{V}$ is an $(N-K)\times(K+1)$ matrix and admits a solution when $N-K \geq K$ and the rank($\mathbf{V}$) is less than $K+1$, that is, condition [C1]. Therefore (3.15) can be used to find, up to a normalization, the $K+1$ unknowns p_k which lead to the K locations t_k. Once the K locations t_k are determined the weights of the Diracs c_k are found by solving the system in (3.9) for $n = 0, \ldots, K-1$. Since $t_k \neq t_l, \forall k \neq l$, the system admits a solution from condition [C2]. ∎

Note that the result does not depend on T. In practice if T is not chosen appropriately then the matrices $\mathbf{V}$ may be ill-conditioned. Figure 3.2(a) illustrates the conditioning of the matrix $\mathbf{V}$ is the least for T close to 0.5 and that the matrix $\mathbf{A}$ is well-conditioned on average.



(a) (b)

Figure 3.2: (a) Average condition number of the matrix that leads to the locations of the Diracs, $\mathbf{V}$, versus the sampling interval T, optimal $T \approx 0.5$; (b) Average condition number of the matrix that leads to the weights of the Diracs, $\mathbf{A}$, versus the sampling interval T, optimal $T \approx 1$. Average is taken on 100 signals with 8 Diracs uniformly spread in the interval $[0, 8]$.

By choosing more adequately the interpolating polynomials, for example by taking the Lagrange polynomials, we may reduce the conditioning of the matrix $\mathbf{V}$, but this remains to be investigated. The algorithm is as follows:

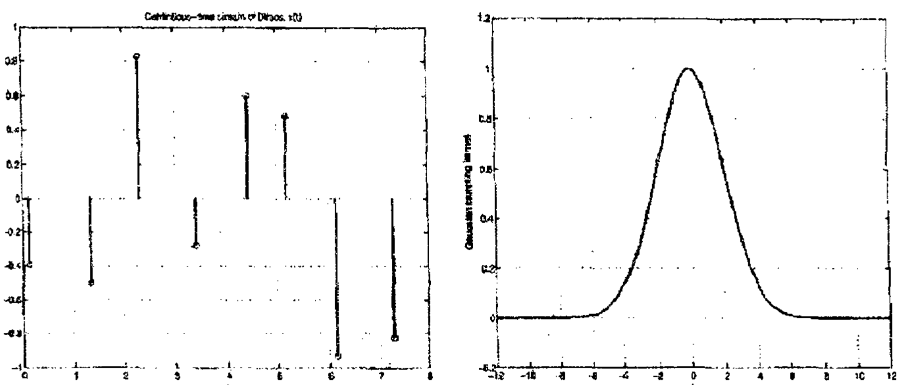
Algorithm 3.1 *Finite length stream of Diracs sampled with a sinc sampling kernel*

Given $y_n = < x(t), \mathrm{sinc}(t/T - n) >, \quad n = 0, 1, \ldots, N-1;$
Calculate $v_{nk} = \Delta^K\left((-1)^n n^k y_n\right), \quad n = K, \ldots, N-1, \quad k = 0, \ldots, K;$
Solve the linear system $\mathbf{V}\cdot\mathbf{p} = 0 \longrightarrow \{p_0, p_1, \ldots, p_K\};$
Find the K roots of $P(u) = \sum_{k=0}^{K} p_k u^k \longrightarrow \{t_0/T, t_1/T, \ldots, t_{K-1}/T\};$
Calculate $a_{nk} = \frac{\sin\left(\pi t_k/T\right)}{\pi\left(t_k/T - n\right)}, \quad n = 0, 1, \ldots, N-1\ ;$
Calculate $Y_n = (-1)^n P(n)\, y_n, \quad n = 0, 1, \ldots, N-1;$
Solve the linear system $\mathbf{A}\cdot\mathbf{c} = \mathbf{Y} \longrightarrow \{c_0, c_1, \ldots, c_{K-1}\}.$

This method can be extended to piecewise polynomials, similarly to Theorem. 2.2. Also, there is an obvious equivalent for discrete-time signals in $\ell^2(\mathbb{Z})$ and discrete-time sinc kernels.

3.1.2 Gaussian sampling kernel

Consider sampling the same signal as in (3.1) but this time with a Gaussian sampling kernel, $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}$, see Figure 3.3. Similar to the sinc sampling



(a) (b)

Figure 3.3: (a) Example of a finite length continuous-time stream of $K = 8$ Diracs randomly spread on an interval $[0, \tau]$ with $\tau = 8$; (b) Gaussian sampling kernel, $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}, \sigma = 2$.

kernel, the samples are obtained by filtering the signal with a Gaussian kernel. Since there are $2K$ unknown variables we show next that N samples with $N \geq 2K$ are sufficient to represent the signal.

Theorem 3.2 *Given a finite stream of K weighted Diracs and a Gaussian sampling kernel $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}$. If $N \geq 2K$ then the N sample values*

$$y_n = < x(t), \varphi_\sigma(t/T - n) > \tag{3.17}$$

are sufficient to reconstruct the signal.

Proof: The sample values are given by

$$\begin{aligned} y_n &= < x(t), e^{-(t/T-n)^2/2\sigma^2} >, \quad n = 0, \ldots, N-1 && (3.18)\\ &= \int_{-\infty}^{\infty} \sum_{k=0}^{K-1} c_k\, \delta(t - t_k)\, e^{-(t/T-n)^2/2\sigma^2}\, dt && (3.19)\\ &= \sum_{k=0}^{K-1} c_k\, e^{-(t_k/T-n)^2/2\sigma^2}. && (3.20) \end{aligned}$$

We expand (3.20) and regroup the terms so as to have variables that depend solely on n and solely on k. We obtain

$$y_n = \sum_{k=0}^{K-1} \left(c_k\, e^{-t_k^2/2\sigma^2 T^2}\right) \cdot e^{nt_k/\sigma^2 T} \cdot e^{-n^2/2\sigma^2} \tag{3.21}$$

which is equivalent to

$$Y_n = \sum_{k=0}^{K-1} a_k\, u_k^n \tag{3.22}$$

where we let $Y_n = e^{n^2/2\sigma^2}\, y_n$, $a_k = c_k\, e^{-t_k^2/2\sigma^2 T^2}$ and $u_k = e^{t_k/\sigma^2 T}$. Note that we reduced the expression Y_n to a linear combination of real exponentials. This hints that the annihilating filter method described in the Section 2.1.1 seems appropriate to find the K values u_k. Let $H(z) = h_0 + h_1\, z^{-1} + \cdots + h_K\, z^{-K}$ be an annihilating filter, that is, h is such that

$$\mathbf{h} * \mathbf{Y} = 0 \tag{3.23}$$

$$\Leftrightarrow \sum_{k=0}^{K} h_k\, Y_{n-k} = 0, \quad n = K, \ldots, N-1. \tag{3.24}$$

Note that this is a Toeplitz system with real exponential components $Y_n = e^{n^2/2\sigma^2}\, y_n$ and therefore a solution exists when the number of equations is greater than the number of unknowns, that is, $N - K \geq K$ and the rank of the system is less than $K+1$ which is the case by hypothesis. Furthermore σ must be carefully chosen otherwise the system is ill-conditioned. If we factor $H(z) = \prod_{k=0}^{K-1} (1 - z^{-1} u_k)$ then we obtain the locations of the Diracs t_k from the roots of the polynomial $H(z)$, that is,

$$t_k = \sigma^2 T \ln u_k. \tag{3.25}$$

Once the values of the Diracs t_k are obtained then we solve for a_k the Vandermonde system in (3.22) for which a solution exists since $u_k \neq u_l, \forall k \neq l$. The weights of the Diracs are simply given by

$$c_k = a_k\, e^{t_k^2/2\sigma^2 T^2}. \tag{3.26}$$

■

Here unlike in the sinc case, we have an almost local reconstruction because of the exponential decay of the Gaussian sampling kernel which brings us to the next topic.

Algorithm 3.2 →

3.2 Infinite length signals with finite local rate of innovation

In this section we consider the dual problem of Sec. 3.1, that is, *infinite* length *signals* $x(t), t \in \mathbb{R}^+$ with a finite *local* rate of innovation and sampling kernels

with *compact support*. In particular, the β-splines of different degree d are considered [73]

$$\varphi_d(t) = (\varphi_{d-1} * \varphi_0)(t), \quad d \in \mathbb{N}^+ \tag{3.27}$$

where $\varphi_0(t)$ is the box spline defined by

$$\varphi_0(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{else} \end{cases} . \tag{3.28}$$

We develop local reconstruction algorithms which depend on moving intervals equal to the size of the support of the sampling kernel.[3] The advantage of local reconstruction algorithms is that their complexity does not depend on the length of the signal. We begin by considering bilevel signals, followed by piecewise polynomial signals.

3.2.1 Bilevel signals

Consider an infinite length continuous-time signal $x(t), t \in \mathbb{R}^+$ which takes on two values, 0 and 1, with initial condition $x(t)\big|_{t=0} = 1$ with a finite local rate of innovation, ρ. These are called bilevel signals and are completely represented by their transition values t_k. For example, binary signals such as amplitude or position modulated pulses or PAM, PPM signals [31], see Fig. 3.4.

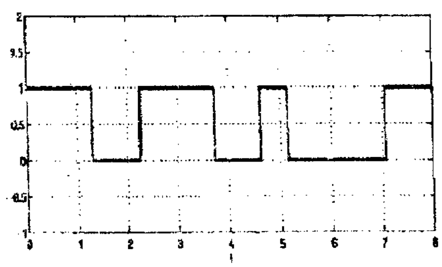


Figure 3.4: Bilevel signal.

Suppose a bilevel signal is sampled with a box spline $\varphi_0(t/T)$. Then the sample values are given by the inner products between the bilevel signal and the box function,

$$y_n = < x(t), \varphi_0(t/T - n) > = \int_{-\infty}^{\infty} x(t)\, \varphi_0(t/T - n)\, dt. \tag{3.29}$$

It can bee seen in Fig. 3.5 that the sample value y_n corresponds to the area occupied by the signal in the interval $[nT, (n+1)T]$. Thus if there is at most one transition per box then we can recover the transition from the sample. This leads us to

[3] The size of the support of $\varphi_d(t/T)$ is equal to $(d+1)T$.

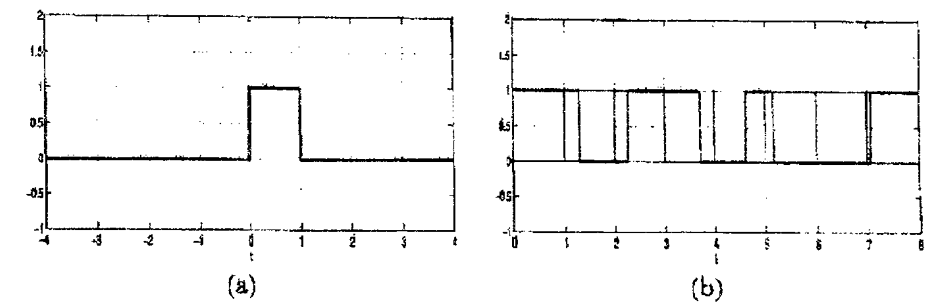


Figure 3.5: (a) Box spline sampling kernel, $\varphi_0(t)$, $T = 1$. (b) Bilevel signal sampled with the box sampling kernel.

Proposition 3.1 *A bilevel signal $x(t), t > 0$, with initial condition $x(t)\big|_{t=0} = 1$, is uniquely determined from the samples $y_n = < x(t), \varphi_0(t/T - n) >$ where $\varphi_0(t)$ is the box spline defined in (3.28) if and only if there is at most one transition t_k in each interval $[nT, (n+1)T]$.*

Proof: For simplicity let $T = 1$. Consider an interval $[n, n+1]$ and suppose $x(n) = 1$. First we show sufficiency followed by necessity .

$\Leftarrow$: If there are 0 transitions in the interval $[n, n+1]$ then the area under the bilevel signal, or the sample value, is $y_n = 1$ since we supposed that $x(t)\big|_{t=n} = 1$. If there is one transition in $[n, n+1]$ then the sample value is equal to

$$y_n = < x(t), \varphi_0(t-n) > = \int_{-\infty}^{\infty} x(t)\, \varphi_0(t-n)\, dt \tag{3.30}$$

$$= \int_{n}^{n+1} x(t)\, dt = \int_{n}^{t_k} 1\, dt = t_k - n \tag{3.31}$$

This implies that $t_k = y_n - n$. Similarly if $x(n) = 0$ then we have $t_k = n + 1 - y_n$. Therefore we can uniquely determine the signal in the interval $[n, n+1]$.

$\Rightarrow$: Necessity is shown by counterexample.
Suppose $x(n) = 1$ and there are two transitions t_k, t_{k+1} in the interval $[n, n+1]$ then the sample value is equal to

$$y_n = \int_{n}^{n+1} x(t)\, dt = \int_{n}^{t_k} 1\, dt + \int_{t_{k+1}}^{n+1} 1\, dt \tag{3.32}$$

$$= t_k - n + n + 1 - t_{k+1} = t_k - t_{k+1} + 1. \tag{3.33}$$

That is, there is one equation with two unknowns and therefore insufficient samples to determine both transitions. Thus there must be at most one transition in an interval $[n, n+1]$ to uniquely define the signal.

■

Now consider shifting the bilevel signal by an unknown shift ϵ, see Fig. 3.6, then there will be two transitions in an interval of length T and one box function will not be sufficient to recover the transitions. Suppose we double the sampling

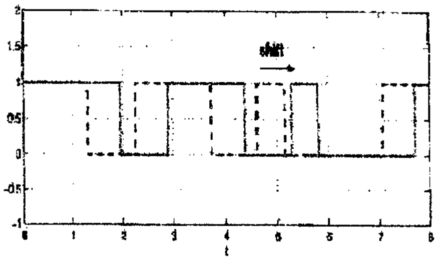


Figure 3.6: Shifted bilevel signal with two transitions in the interval [5, 6].

rate, then the support of the box sampling kernel is doubled and we have two sample values y_n, y_{n+1} covering the interval $[nT, (n+1)T]$ but these values are identical (see their areas). Therefore increasing the sampling rate is still insufficient.
This brings us to consider a sampling kernel not only with a larger support but with added information. For example, the hat spline function $\varphi_1(t/T)$ defined by

$$\varphi_1(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{else} \end{cases} \tag{3.34}$$

leads to sample values defined by $y_n = < x(t), \varphi_1(t/T - n) >$ or

$$y_n = \int_{(n-1)T}^{nT} x(t)\left(1 + t/T - n\right) dt + \int_{nT}^{(n+1)T} x(t)\left(1 - (t/T - n)\right) dt. \tag{3.35}$$

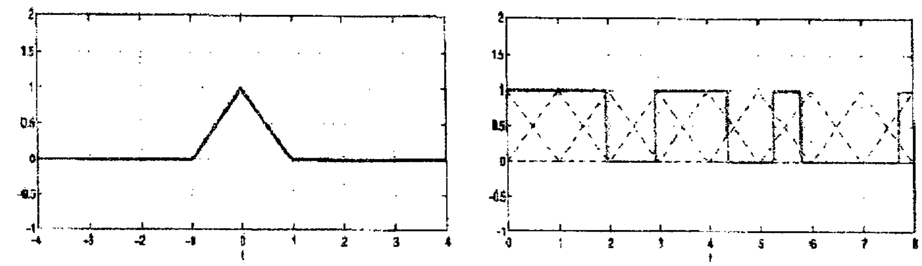
From Fig. 3.7 we can see that there are two sample values covering the interval $[nT, (n+1)T]$. We will show next that in this case we can uniquely determine the signal.

Proposition 3.2 *An infinite length bilevel signal $x(t)$, with initial condition $x(0) = 1$ is uniquely determined from the samples defined by*

$$y_n = < x(t), \varphi_1(t/T - n) > \tag{3.36}$$

where $\varphi_1(t)$ is the hat sampling kernel if and only if there are at most two transitions $t_k \neq t_j$ in each interval $[nT, (n+2)T]$.

Proof: Again, for simplicity let $T = 1$ and suppose the signal is known for $t \le n$ and $x(t)\big|_{t=n} = 1$.
First we show sufficiency by showing the existence and uniqueness of a solution. Then we show necessity by a counterexample.



(a) (b)

Figure 3.7: (a) Hat spline sampling kernel,$\varphi(t/T)$, $T=1$. (b) Bilevel signal with two transitions in an interval $[n, n+1]$ sampled with a hat sampling kernel.

$\Leftarrow$: Similar to the box sampling kernel the sample values will depend on the configuration of the transitions in the interval $[n, n+2]$. If there are at most 2 transitions in the interval $[n, n+2]$ then the possible configurations are

$$(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)$$

where the first and second component indicate the number of transitions in the intervals $[n, n+1]$, $[n+1, n+2]$ respectively, see Fig. 3.8. Furthermore since the hat sampling kernel is of degree one we obtain for each configuration a quadratic system of equations with variables t_0, t_1.

$$y_n = \int_{n-1}^{n} x(t)(1+t-n)\,dt + \int_{n}^{n+1} x(t)(1-(t-n))\,dt \tag{3.37}$$

$$y_{n+1} = \int_{n}^{n+1} x(t)(1+t-(n+1))\,dt + \int_{n+1}^{n+2} x(t)(1-(t-(n+1)))\,dt. \tag{3.38}$$

First we show that the quadratic system of equations admits a solution and then that it is unique.

(a) Existence.
Take $n=0$ and so the moving interval is $[0,2]$.
The configuration $(0,0)$ will lead to sample values $y_0 = 1$, $y_1 = 1$.
The configuration $(0,1)$ will lead to sample values

$$y_0 = 1/2 + \int_1^{t_0}(t-1)\,dt = \frac{1}{2}{t_0}^2 + 1 - t_0 \tag{3.39}$$

$$y_1 = 1/2 + \int_1^{t_0}(2-t)\,dt = -\frac{1}{2}{t_0}^2 - 1 + 2t_0 \tag{3.40}$$

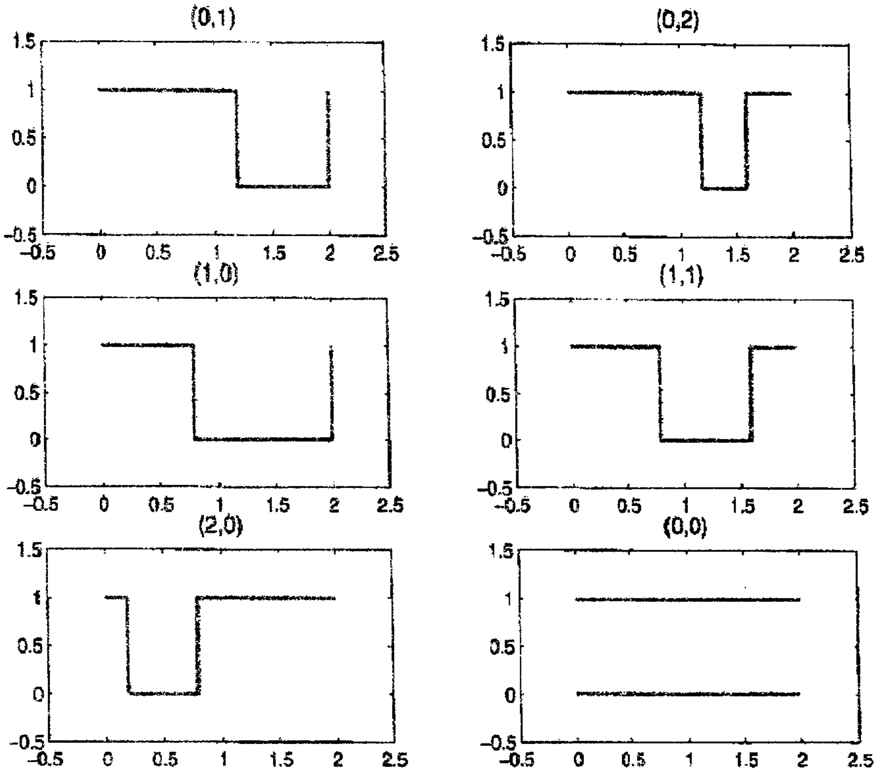


Figure 3.8: Bilevel signal containing at most 2 transitions in the interval $[0,2]$: All possible configurations.

$\Rightarrow t_0 = y_0 + y_1 = 1 + \sqrt{-1+2y_0} = 2 - \sqrt{2-2y_1}$.
The configuration $(0,2)$ will lead to sample values

$$y_0 = \frac{1}{2}{t_0}^2 + 1 - t_0 + t_1 - \frac{1}{2}{t_1}^2 \tag{3.41}$$

$$y_1 = -\frac{1}{2}{t_0}^2 + 1 + 2t_0 + \frac{1}{2}{t_1}^2 - 2t_1 \tag{3.42}$$

$\Rightarrow t_0 = \frac{(-2-2y_1+y_0^2+2y_0y_1+y_1^2)}{2(-2+y_0+y_1)}, t_1 = \frac{-(10-6y_1-8y_0+y_1^2+y_0^2+2y_0y_1)}{2(-2+y_0+y_1)}$.
The configuration $(1,0)$ will lead to sample values

$$y_0 = -\frac{1}{2}{t_0}^2 + t_0 \tag{3.43}$$

$$y_1 = \frac{1}{2}{t_0}^2 \tag{3.44}$$

$\Rightarrow t_0 = y_0 + y_1 = 1 - \sqrt{1-2y_0} = \sqrt{2y_1}$.

The configuration (1, 1) will lead to sample values

$$y_0 = -\frac{1}{2}\,t_0{}^2 + t_0 - \frac{1}{2}\,t_1{}^2 + t_1 \tag{3.45}$$

$$y_1 = \frac{1}{2}\,t_0{}^3 + 2 + \frac{1}{2}\,t_1{}^2 - 2\,t_1 \tag{3.46}$$

$\Rightarrow t_0 = \frac{y_0+y_1+\sqrt{-y_1^2-2y_0y_1+4y_1-y_0^2}}{2}, t_1 = \frac{-y_1-y_0+4+\sqrt{-y_1^2-2y_0y_1+4y_1-y_0^2}}{2}$

The configuration (2, 0) will lead to sample values

$$y_0 = -\frac{1}{2}\,t_0{}^2 + 1 + t_0 + \frac{1}{2}\,t_1{}^2 - t_1 \tag{3.47}$$

$$y_1 = \frac{1}{2}\,t_0{}^2 + 1 - \frac{1}{2}\,t_1{}^2 \tag{3.48}$$

$\Rightarrow t_0 = \frac{2-2y_1+y_1^2+2y_0y_1+y_0^2-4y_0}{2(-2+y_1+y_0)}, t_1 = -\frac{6-4y_0-6y_1+y_1^2+2y_0y_1+y_0^2}{2(-2+y_1+y_0)}.$

(b) Uniqueness.
If $y_n = 1$ and $y_{n+1} = 1$ then this implies configuration (0, 0) .
If $y_n = 1$ and $1/2 \leq y_{n+1} \leq 1$ then the possible configurations are (0, 1), (0, 2). By hypothesis, there are at most two transitions in the interval $[n+1, n+3]$ therefore if $y_{n+2} \leq 1/2$ then the configuration in the interval $[n, n+2]$ is (0, 1) otherwise if $y_{n+2} \geq 1/2$ then the configuration is (0, 2).
If $1/2 \leq y_n \leq 1$ and $1/2 \leq y_{n+1} \leq 1$ then this implies configuration (2, 0).
If $1/2 \leq y_n \leq 1$ and $0 \leq y_{n+1} \leq 1/2$ then this implies configuration (1, 0).

$\Rightarrow$: Necessity is shown by counterexample.
Consider a bilevel signal with three transitions in the interval [0, 2] but with all three in the interval [0, 1], see Fig. 3.9. Then the sample values in this case are are equal to

$$y_0 = 1/2 + \int_0^{t_0} (1-t)\,dt + \int_{t_1}^{t_2} (1-t)\,dt \tag{3.49}$$

$$= 1/2 + t_0 - t_1 + t_2 - t_0^2/2 + t_1^2/2 - t_2^2/2 \tag{3.50}$$

$$y_1 = \int_0^{t_0} t\,dt + \int_{t_1}^{t_2} t\,dt \tag{3.51}$$

$$= t_0^2/2 - t_1^2/2 + t_2^2/2. \tag{3.52}$$

There is no unique solution for this quadratic system of equations. Therefore there must be at most 2 transitions in an interval [0, 2].

■

Once again if there is an unknown shift in the bilevel signal then there may be three transitions in an interval $[nT, (n+1)T]$ and so we increase the number of samples by sampling with $\varphi_1(t/(T/2))$. The pseudo-code for sampling bilevel signals using the box and hat functions are given in full detail in Section 3.2.3. When going to higher order splines, necessity carries over. Sufficiency is more tedious since we must solve a system of higher order polynomial equations.

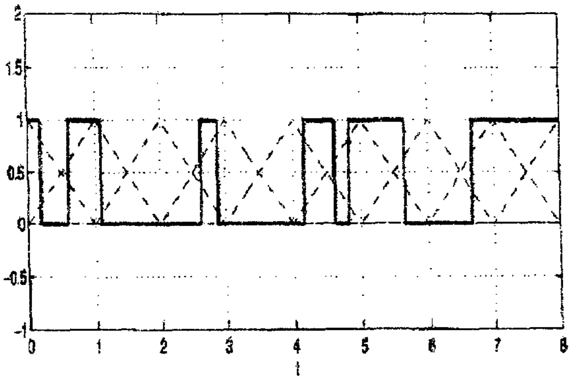


Figure 3.9: Bilevel signal containing three transitions in an interval $[4, 5]$, sampled with the hat sampling kernel $\varphi_1(t)$.

3.2.2 Piecewise polynomials

Similar to bilevel signals we consider sampling piecewise polynomials with the box sampling kernel. Consider an infinite length piecewise polynomial signal $x(t)$ where each piece is a polynomial of degree R and defined on an interval $[t_{k-1}, t_k]$, that is,

$$x(t) = \begin{cases} x_0(t) = \sum_{m=0}^{R} c_{0m}\, t^m & t \in [0, t_0] \\ x_1(t) = \sum_{m=0}^{R} c_{1m}\, t^m & t \in [t_0, t_1] \\ \vdots & \\ x_K(t) = \sum_{m=0}^{R} c_{Km}\, t^m & t \in [t_{K-1}, t_K] \\ \vdots & \end{cases} \qquad (3.53)$$

Each polynomial piece $x_k(t)$ contains $R+1$ unknown coefficients c_{km}. The transition value t_k is easily obtained once the pieces $x_{k-1}(t)$ and $x_k(t)$ are determined, thus there are $2(R+1)+1$ degrees of freedom. If there is one transition in an interval of length T the maximal local rate of innovation is $\rho_m(T) = (2(R+1)+1)/T$. Therefore in order to recover the polynomial pieces and the transition we need to have at least $2(R+1)+1$ samples per interval T. This is achieved by sampling with the following box sampling kernel $\varphi_0(t/\frac{T}{2(R+1)+1})$. For example if $x(t)$ is a piecewise linear signal with 2 pieces as illustrated in Fig. 3.10 then to recover the signal it is sufficient to take 5 samples: two before the transition, two after the transition and one sample covering the transition.

We can generalize by noting that the Rth derivative of a piecewise polynomial of degree R is a piecewise constant signal. The pseudo-code for sampling piecewise

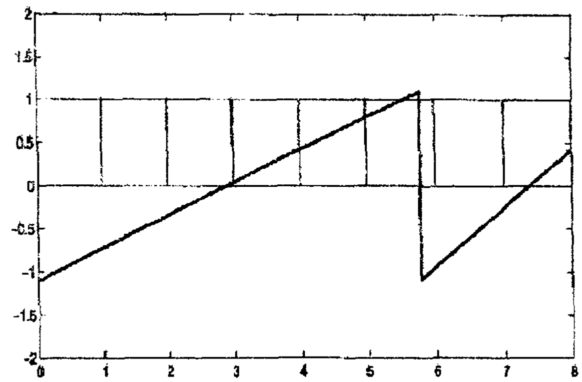


Figure 3.10: Piecewise linear signal sampled with a box sampling kernel.

constant signals with the box sampling kernel is found in Section 3.2.3 .

3.2.3 Local reconstruction algorithms

The following algorithms have been implemented in Maple^{TM}. In all of the algorithms k is the index of a transition value and n is the index of the current interval $[n, n+1]$. We suppose that $x(t) = 1, \forall t \leq 0$.

Bilevel signals

Suppose N sample values $y_n = < x(t), \varphi_0(t-n) >$ are available.

Algorithm 3.~~2~~3 *Bilevel signal with box sampling kernel.*

Require: $k \leftarrow 0, n \leftarrow 0$
  while $n \leq N-1$ ***do***
    if $y_n = 1$ ***then***
      $x(t) = 1 \quad \forall t \in [n, n+1]$.
    end if
    if $y_n = 0$ ***then***
      $x(t) = 0 \quad \forall t \in [n, n+1]$.
    end if
    if $0 < y_n < 1$ ***then***
      if $x(n^+) = 1$ ***then***
        $t_k \leftarrow y_n + n$
      else
        $t_k \leftarrow n + 1 - y_n$
      end if
      $k \leftarrow k+1$
    end if
    $n \leftarrow n+1$
  end while

Next we give the pseudo-code for bilevel signals sampled with a hat sampling kernel. Suppose N sample values $y_n = < x(t), \varphi_1(t-n) >$ are available. The variable tncode is a set whose last component indicates the number of transitions in the interval $[n-1, n]$.

Algorithm 3.~~2~~ 4 *Bilevel signal with hat sampling kernel.*

Require: *tncode* $\leftarrow \emptyset$, $k \leftarrow 0$, $n \leftarrow -1$
while $n \leq N-1$ ***do***
if $y_n = 1$ ***then***
tncode $\leftarrow 0$
$x(t) = 1 \qquad \forall t \in [n-1, n+1]$
end if
if $y_n = 0$ ***then***
tncode $\leftarrow 0$
$x(t) = 0 \qquad \forall t \in [n-1, n+1]$
end if
if $0 < y_n < 1$ ***then***
if 0 *transitions in the interval* $[n-1, n]$ ***then***
if $y_n = 0.5$ ***then***
$t_k \leftarrow n$
tncode $\leftarrow 0$
else
sol = *solve for configuration* $(1,0) \in [n, n+2]$
if *sol* $\neq \emptyset$ ***then***
tncode $\leftarrow 1, 0$
if $x(n^+) = 1$ ***then***
$t_k \leftarrow n + 1 - \sqrt{2 - 2y_n}$
else
$t_k \leftarrow n + 1 - \sqrt{2}\sqrt{y_n}$
end if
$n \leftarrow n + 2$
$k \leftarrow k + 1$
solfound $\leftarrow$ *True*
else
sol = *solve for configuration* $(1,1) \in [n, n+2]$
if *sol* $\neq \emptyset$ ***then***
tncode $\leftarrow 1, 1$
if $x(n^+) = 1$ ***then***
$t_k \leftarrow n + 1 - \sqrt{2 - 2y_n}$
$t_{k+1} \leftarrow n + 2 - \sqrt{-3 + 2y_{n+1} + 2\sqrt{2 - 2y_n} + 2y_n}$
else
$t_k \leftarrow n + 1 - \sqrt{2}\sqrt{y_n},$
$t_{k+1} \leftarrow n + 2 - \sqrt{1 + 2\sqrt{2}\sqrt{y_n} - 2y_{n+1} - 2y_n}$
end if
$n \leftarrow n + 2$
$k \leftarrow k + 2$
solfound $\leftarrow$ *True*
else
solfound $\leftarrow$ *False*

end if
end if
end if
if *not solfound* ***then***
sol = *solve for configuration* $(2,0) \in [n, n+2]$
if *sol* $\neq \emptyset$ ***then***
tncode $\leftarrow 2,0$
if $x(n^+) = 1$ ***then***
$t_k \leftarrow \frac{1}{2}\,\frac{2-2\,y_{n+1}-4\,y_n+2\,y_{n+1}\,y_n+y_n^2+y_{n+1}^2-4\,n+2\,y_n\,n+2\,y_{n+1}\,n}{-2+y_{n+1}+y_n}$
$t_{k+1} \leftarrow -\frac{1}{2}\,\frac{6-6\,y_{n+1}+4\,n-4\,y_n-2\,y_n\,n+y_n^2+2\,y_{n+1}\,y_n-2\,y_{n+1}\,n+y_{n+1}^2}{-2+y_{n+1}+y_n}$
else
$t_k \leftarrow -\frac{1}{2}\,\frac{-2\,y_{n+1}-2\,y_n\,n+y_n^2+2\,y_{n+1}\,y_n-2\,y_{n+1}\,n+y_{n+1}^2}{y_n+y_{n+1}}$
$t_{k+1} \leftarrow \frac{1}{2}\,\frac{2\,y_{n+1}+2\,y_n\,n+y_n^2+2\,y_{n+1}\,y_n+2\,y_{n+1}\,n+y_{n+1}^2}{y_n+y_{n+1}}$
end if
$n \leftarrow n+2$
$k \leftarrow k+2$
end if
end if
else if 1 *transition in the interval* $[n-1, n]$ ***then***
sol = *solve for configuration* $(1,0) \in [n-1, n+1]$ *given* $t_{k-1} \in [n-1, n]$
if *sol* $\neq \emptyset$ ***then***
tncode $\leftarrow 0$
$n \leftarrow n+1$
else
sol = *solve for configuration* $(1,1) \in [n-1, n+1]$ *given* $t_{k-1} \in [n-1, n]$
if *sol* $\neq \emptyset$ ***then***
tncode $\leftarrow 1$
if $x((n-1)^+) = 1$ ***then***
$t_k \leftarrow n + \sqrt{2-2\,y_{n+1}}$
else
$t_k \leftarrow n + \sqrt{2}\,\sqrt{y_{n+1}}$
end if
$n \leftarrow n+1$
$k \leftarrow k+1$
end if
end if
else
{ 2 *transitions in the interval* $[n-1, n]$}
tncode $\leftarrow 0$
end if
end if
end while

Piecewise constant signal

We consider sampling a piecewise constant signal with the box sampling kernel. doubling the sampling rate is sufficient to recover the signal, thus we suppose

$2N$ sample values y_n are available.

Algorithm 3.~~1~~5 *Piecewise constant signal with box sampling kernel.*

Require: , $k \leftarrow 0, n \leftarrow 0$ $y_n, n = 0 \ldots 2N-1$

while $n \leq 2N-2$ ***do***

  if $|y_{n+1} - y_n| = 0$ ***then***

    $n = n + 1$

  else

    $c_k = y_{n-1}$

    $c_{k+1} = y_{n+1}$

    $t_k = \frac{y_n + nc_k - (n+1)c_{k+1}}{c_k - c_{k+1}}$

    $n \leftarrow n + 2$

    $k \leftarrow k + 1$

  end if

end while

3.3 Summary

- A finite length stream of K Diracs can be recovered from N samples y_n obtained as the inner product between the signal and shifted versions of the sinc and Gaussian sampling kernel, when $N \geq 2K$.
- For both types of sampling kernels two systems of equations must be solved: the first system is to find the locations of the Diracs and the second is to find the weights of the Diracs.
- When sampling a randomly spaced stream of Diracs with the *sinc* kernel the system leading to the transitions may be ill-conditioned if the sampling interval T is not chosen appropriately. It is illustrated that at critical sampling, that is, when we have $N = 2K$ sample values, the optimal sampling interval obtained for these type of signals is $T = 0.5$.
- When sampling a randomly spread stream of Diracs with the *Gaussian* kernel the conditioning of both systems depends also on the value of the variance σ^2 in the Gaussian kernel.
- In the same fashion as in Chapter 2, the sampling schemes using the sinc and the Gaussian kernels can be generalized to both continuous-time and discrete-time piecewise polynomial signals.
- Infinite length signals were sampled using a compact support sampling kernel.
- Bilevel signals can be recovered using a *Box* sampling kernel $\varphi_0(t/T)$ if and only if there is at most one transition in each interval $[n, (n+1)T]$.
- Bilevel signals can be recovered using a *Hat* sampling kernel $\varphi_1(t/T)$ if and only if there is at most two transitions in each interval $[n, (n+2)T]$.
- In general, to recover the infinite length piecewise polynomials with K pieces of of maximum degree R using a box sampling kernel, the sampling rate must be greater than the maximum local rate of innovation

$$\rho_m(T) = (2(R+1)+1)/T.$$

- Sampling and reconstruction algorithms were given for each problem in their respective sections.

Sampling of Piecewise Polynomial Signals

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December 14, 1999

Abstract

In this note, we consider the problem of sampling piecewise polynomial signals using smooth sampling kernels. A few results and a number of conjectures are shown regarding the possibility to exactly reconstruct piecewise polynomial signals from appropriate sampling.

1 Introduction

Sampling of bandlimited signals is a standard signal processing procedure. However, many signals in practice are not bandlimited, and bandlimiting them first might distort them unnecessarily.

Bandlimited signals have the property that, unlike general signals, a countable set of parameters is sufficient to specify them (e.g. the Fourier coefficients at the periodized spectrum). There are many other signals besides bandlimited ones that are specified by a countable set of parameters, and we consider hereafter one such class, namely piecewise polynomial signals.

We pose the problem of finding the parameters of certain piecewise polynomial signals through regular sampling with a sampling kernel. Typically, the sampling kernel is a smooth function with a support at least as large as the sampling interval. (e.g. box or hat function).

Thus, we have access to the inner products between the signal and shifted versions of the sampling kernel. The question then is now to recover the exact parameters of the signal.

Figure 1: Bilevel 0/1 signal.

Figure 2: Transition at t_0 and box function.

The applications of these methods can be found in signal processing for communications and sampling of images with special characteristics (e.g. bi-level images).

The algorithms involved involve either the solution of systems of non-linear equations, or iterative solution methods.

These notes are an outline of the problems, some elementary results, a set of conjectures, and possible extensions.

2 An elementary example: bi-level signals and spline kernels

Consider bi-level signals, w.l.o.g $\{0,1\}$ signals, with transitions at arbitrary times, see Fig. 1.

Clearly, they are not bandlimited, and thus sampling with a $\delta(t)$ kernel will not work. Consider sampling at integers using a box spline centered at 0:

$$\varphi_0(t) = \begin{cases} 1 & t \in [-1/2, 1/2] \\ 0 & \text{else} \end{cases}$$

Further, assume that $x(0) = 1$, and that $x(t)$ has at most 1 transition over any interval $[t, t+1], t \in \mathbb{R}$, or that transitions are spaced apart by a distance $d > 1$. Consider the set of samples

$$x[n] =< \varphi_0(t-n), x(t) >$$

then

Result 1: $x[n]$ specifies uniquely $x(t)$ (with some initial conditions on $x(t)$)

Proof: (Sketch) The value of $x[n]$ gives immediately the transition location. Consider $t = 0$. Then $t_0 = x[n] - 1/2$ or $t_0' = -t_0$.

However, there is uncertainty about t_0, t_0'. Thus, we need initial conditions, like $x(t)$ is equal to 1 from $-1/2$ to 0 for example. Once the first t_0 is specified, the other follow similarly.

If there are more transitions, the sampling is not sufficient, e.g. see Fig. 3.

Figure 3: Uncertainty with transitions that are too close.

Figure 4: Shift variance.

Note: The restriction of most 1 transition in any interval $[t, t+1]$ is more than what is really needed. It is sufficient to have at most 1 transition in the intervals $[t-1/2, t+1/2]$ (which is less restrictive). But then the origin of time becomes critical, and the sampling condition is not shift invariant anymore e.g. see Fig. 4

Using a sampling kernel that is of wider spread allows to relax the restriction on the number of transitions: on the one hand, the "average number of transition" or "transition rate λ" cannot exceed 1 (per interval), but on the other hand, it must hold only over the spread of the sampling kernel, not over each length 1 interval. Consider the hat function:

$$\varphi_1(t) = \begin{cases} 1-|t| & |t|<1 \\ 0 & \text{else} \end{cases}$$

and 2 transitions over the $[0,1]$ interval (again assuming $x(t) = 1, t \in [-1,0]$ and $t \in [1,2]$ as boundary conditions).

Then:

$$\begin{aligned} x[0] = <\varphi_1(t), x(t)> &= 1/2 + \int_0^{t_0} (1-t)dt + \int_{t_1}^1 (1-t)dt \\ &= 1/2 + t_0 - t_0^2/2 + 1/2 - t_1 + t_1^2/2 \\ &= 1 + t_0 - t_1 - t_0^2/2 + t_1^2/2 \end{aligned}$$

$$\begin{aligned} x[1] = <\varphi_1(t-1), x(t)> &= 1/2 + \int_0^{t_0} tdt + \int_{t_1}^1 tdt \\ &= 1/2 + t_0^2/2 + 1/2 - t_1^2/2 \\ &= 1 + t_0^2/2 - t_1^2/2 \end{aligned}$$

Note: $x[0] + x[1] = 2 + t_0 + t_1$.

So, by solving this 2 by 2 non linear system, we can find t_0, t_1, from $x[0], x[1]$.

Figure 5: Hat function and two transitions.

Conjecture 1: For the linear spline case, if we have K intervals and at most 2 transitions on intervals of length 2, then the transitions can be computed uniquely (with adequate boundary conditions on the left and right side).

Clearly, more than K transitions cannot be recovered, since we have K measurements only. More than 2 transitions in an interval $[n, n+2]$ should lead to ambiguity (need to be shown, obvious in box spline case).

The above result can certainly be extended to splines of arbitrary order:

Conjecture 2: For splines of order N, there can be at most $N+1$ transitions in intervals $[n, n+N+1]$. If so, the transition times can be exactly recovered.

Remark: as $N \to \infty$ (e.g. Gaussian kernel) we get something like the Kadec $-1/4$ theorem, since the average rate has to be 1.

Note: recovering $\{t_i\}$ becomes increasingly difficult, since it involves NL equ. with t_i^N.

Question 1: What happens if the kernel used in the sampling is the sinc kernel? Following Conjecture 2, we should be able to recover the t_i's just as well. But is there an interesting method, or on the contrary, is the problem absurdly difficult because of long range interaction.

Question 2: Is there an iterative algorithm, in particular one that has good convergence properties? For example, in the example with linear splines, can we "efficiently" search t_0, t_1, e.g.

$$\begin{aligned}
x^{(i)}[0] &= fct[t_0^{(i)}, t_1^{(i)}] && \text{(given by linear splines)}\\
x^{(i)}[1] &= fct[t_0^{(i)}, t_1^{(i)}] && \text{(given by linear splines)}\\
t_0^{(i+1)} &= fct[\Delta_0^{(i)}, \Delta_1^{(i)}, t_0^{(i)}] && \text{(to be found)}\\
t_1^{(i+1)} &= fct[\Delta_0^{(i)}, \Delta_1^{(i)}, t_1^{(i)}] && \text{(to be found)}
\end{aligned}$$

While we know there is a solution that satisfies the measured values, the problems look more like root finding, and might get stuck in local minimas. This needs investigation.

3 Piecewise constant signals

How can we apply the previous ideas to signals that are piecewise constant? Astonishingly, it suffices to double the sampling rate!

Result 2: If there is at most 1 transition in the intervals $[n, n+2]$, then sampling with the box spline is sufficient to reconstruct $x(t)$.

Proof: (sketch) There are technicalities to set up ($x(t)$: boundaries?) (e.g. when the signal is zero versus non-zero). Also, requiring the interval

Figure 6: Piecewise constant signals.

Figure 7: Multilevel signal.

to be a running interval $(t, t+2)$ might make it simpler. In any case, one of the box splines "measures" the value, while the other derives the transition point: For example see Fig. 6

$$V_0 =< \varphi_0(t), x(t) >$$
$$V_1 =< \varphi_0(t-1), x(t) >$$
$$V_2 =< \varphi_0(t-2), x(t) >$$

Then: t_0 can be derived from V_1 and V_0, V_2.

Conjecture 3: The above result can be extended to arbitrary spline kernels, with necessary adjustments.

In that case, the same "relaxation" happens as in the bi-level signal, that is, only a bigger interval $(t, t+2N)$ enters into the picture.

In this multilevel case, we can consider 2 specific cases.

i) Multilevel with specific levels, e.g. the constant pieces have integer values (See Fig. 7.) This should lead to potential simplifications, but prob. not to a lowering of the sampling rate necessary to reconstruct.

ii) Piecewise constant signal with noise, potentially with noise levels dependent on value. This is more realistic, and more difficult. A detailed analysis on error propagation is necessary. (See Fig. 8).

4 Piecewise Polynomial Signals

Take the simplest case:

i) signals are piecewise linear and continuous

ii) the sampling kernel is the box spline

Then, with sufficient oversampling, one should be able to recover the signal.

Figure 8: Multilevel signal with level dependent noise.

Figure 9: Piecewise linear signal, and two box splines measuring a linear piece.

Figure 10: Piecewise linear signal, where discontinuities can be found.

Conjecture 4: If there are less than 1 transition per interval of length 4, then the piecewise linear signal can be recovered from box spline sampling.

Sketch of idea : see Fig. 9.

There are at least 2 splines inside each linear piece. These allow to measure the linear piece exactly. However, we are not sure where the breaks are. With the restriction above, there will always be 3 box splines on a linear piece, thus leading to $x[i], x[i+1], x[i+2]$ that are discrete-time linear. This allows to test where the breaks are ($x[k]$ which is not aligned): see Fig. 10.

Here: $[i, i+1, i+2]$ are aligned, but $[i-1, i, i+1]$ and $[i+1, i+2, i+3]$ are not.

Conjecture 5: Using higher order splines relaxes the transition to be within a longer interval.

Conjecture 6: Similar results can be developed for piecewise polynomials of higher degree, with singularities (discontinuity). Appropriate "oversampling" is needed. (Enough sampling kernels "inside" to specify the piece).

For example, the method developed in Conjecture 4 allows piecewise linear signals with discontinuity, since each linear piece is specified independently. Then, the sample of the overlap ($x[i-1]$ in the example) can be used to specify the discontinuity, or in a diagram. See Fig. 11 (a).

The value of the "overlap" sample specifies if the transition is as in Fig. 11 (b).

With these results, we have a "sampling theory" for piecewise polynomial signals.

5 Extension to two dimensions

Clearly, a very interesting case appears in two dimensions, where bi-level images are quite common (e.g. the woodcut picture).

The two-dimensional case is however much more involved. Singularities are now of dimension 1 (rather than 0), and so cannot measured through sampling, unless they are themselves restricted.

Figure 11: (a) Finding the discontinuity location (b) Finding the exact value.

Figure 12: (a) Arbitrary boundary (b) Smooth boundary.

Figure 13: Bandlimited boundary function.

Consider the following two cases in Fig. 12:

- (a) Unit square, with 0/1 transition given by arbitrary function and
- (b) Unit square, with 0/1 transition given by smooth (e.g. "bandlimited") function.

Clearly, the method developed in the previous sections can be applied on a set of lines (e.g. a square grid), and it will work for case (b), but not for case (a) a priori.

Conjecture 7: By restricting the transition or singularity to be bandlimited, one can give an exact sampling using separable spline kernels.

Sketch of obvious case: assume a boundary function that is a bandlimited function (not a curve, but a function), that is, on an infinite image, see Fig. 13.

Apply the spline method on vertical lines, at sampling intervals $T < \pi/\Omega_{max}$. This gives exact samples of the boundary function, which can now by perfectly interpolated using Shannon's theorem.

Similar results can be applied if the boundary function is piecewise polynomial.

Conjecture 8: If the boundary function is piecewise polynomial (with a minimum separation between transitions) there is a sufficiently fine separable sampling using spline kernels that allows a perfect representation of the boundray, and thus of the bi-level image. Example: piecewise linear boundary, see Fig. 14

The last results can most probably be generalized to:

Conjecture 9: If an image is piecewise polynomial with boundaries that are either bandlimited or piecewise polynomial, then separable sampling using spline kernels can give a complete characterization if sampling is fine enough.

Note: a bandlimited assumption on contours is not unreasonable in our example.

Addendum to notes

So far we have considered the two dimensional case in a separable fashion, which is too limited. Instead, it is of interest to consider the sampling of two

Figure 14: Piecewise linear boundary.

Figure 15: Two-dimensional bi-level signal.

Figure 16: Intersection of boundaries

dimensional piecewise polynomial signals, where the sampling kernel is a non-separable point spread function as is the case in a "real" sampling system (e.g. a scanner).

Conjecture 10:

A bi-level two-dimensional signal with piecewise linear boundaries of "sufficient separation"[1] can be perfectly recovered from a sufficiently dense[2] sampling with a separable or non-separable kernel.

The idea of the proof is elementary, however, the various cases and details are tricky. So we only discuss the idea here. Consider the following bi-level two dimensional signal:

For simplicity, we consider the square box function as the sampling kernel (it is separable, but we do not use its separability in its solution).

Call Sij the value of the "sample" corresponding to cell i, j. Assume the boundary is given by $y = ax + b$. Then $x_1 = (1-b)/a, x_2 = (2-b)/a$ and $x_2 - x_1 = 1/a$. Then it follows that

$$S_{00} = (1-b)^2/2a \tag{1}$$

$$S_{01} = (3-b)/2a \tag{2}$$

This leads to a quadratic equation for b

$$b^2 - 2b(1 + S_{00}/S_{01})!1 - 2S_{00}/S_{01} \tag{3}$$

where the positive real root is the solution. Then a follows from 1 or 2.

Clearly things are more complex, since one needs to find two cells on the boundary, but it is also clear that a similar method works. In the case of more general point spread function, numerical integration is needed, and probably a look up table using pairs of values can be used.

What happens if boundaries intersect? If enough cells are found that define the boundaries, then the intersection is defined exactly as shown in Fig. 16.

[1] This notion, which is intuitive, needs to be formalized.

[2] Again, notion needs to be made exact.

In this case, there are enough cells outside the "intersection cell" that defined the boundaries and the intersection. To define exactly the density of cells versus the maximum number of individual boundary elements is one of the open challenges, even if we intuitively see when it will work and when it won't.

6 Relations to other constructions

i) Kadec's $-1/4$ result: there is probably a relation to this result. In particular, as the support of the sampling kernel goes to infinity, only an average rate constraint remains, just as in Kadec's result on irregular sampling

ii) usual sinc sampling: we have only hinted of the relation, but there is a connection, and again, since the support of sinc is infinite, we probably get an "average rate" result. However, sinc based sampling is much more complex.

iii) Logan's theorem on zero crossings: Again, it is a result on bandlimited signals, so there is no direct connection. However, it might be interesting to see if zero crossings, which define polynomials, can be used to define piecewise polynomials as well. This is unclear, since breakpoints are not zero crossings in general.

iv) Derivative sampling, integral sampling: it is of interest to see if our results carry over to derivative and integral sampling. A priori, considering the bi-level signal case, the signal and its (generalized) derivative carry the same information, so sampling one or the other, or alternatively one and the other, gives the same information.

v) Radon transform: in the context of bi-level images with smooth boundaries, characteristics of the Radon transform need to be looked at (is the Radon transform smooth, can it be sampled etc.).

Conclusion

The above sketched results should serve as a motivation to fully understand this problem.

More generally, we conjecture that any object with a countable set of parameters that are sufficiently spread out can be recovered by sufficiently fine uniform sampling.

F1

$x(t)$:

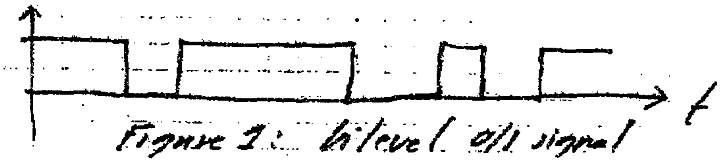


Figure 1: bilevel 0/1 signal

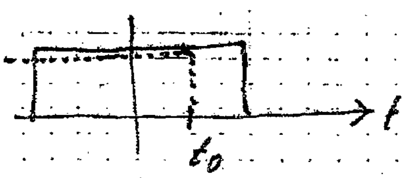


Figure 2: Transition at t_0 of Box function

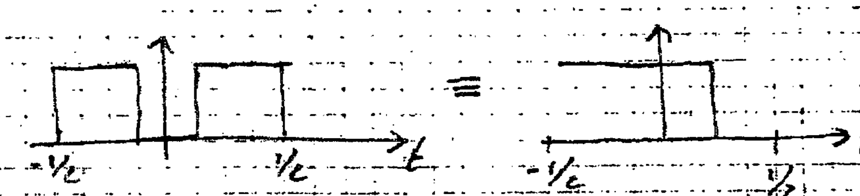


Figure 3: Uncertainty with transitions that are too close.

F2

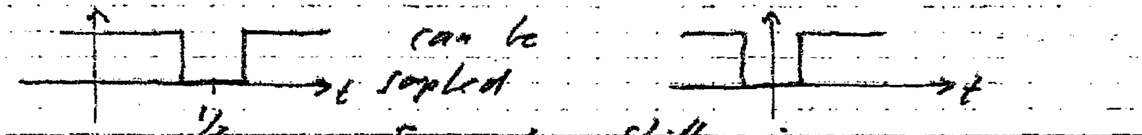


Figure 4: Shift variance

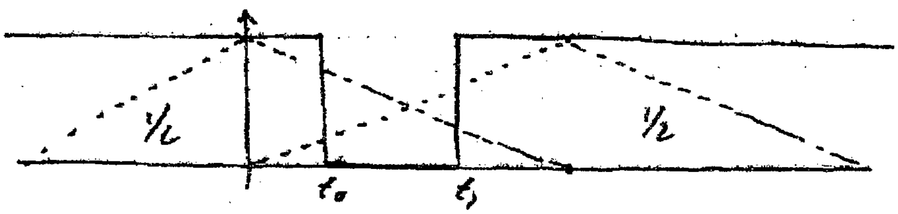
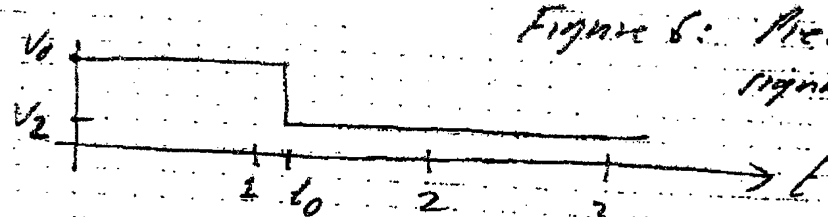


Figure 5: hat function and two transitions.

Figure 6: Piecewise constant signals.



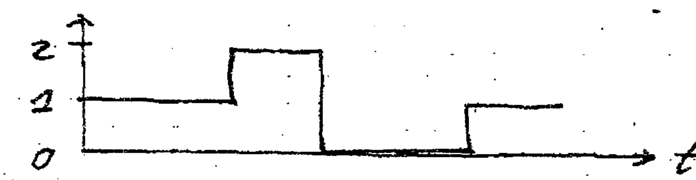


Figure 7: Multilevel signal

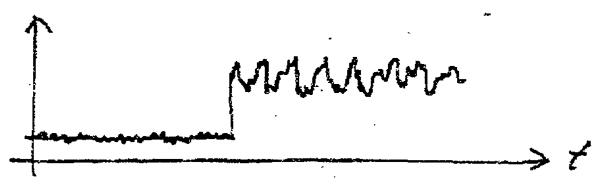


Figure 8: Multilevel signal with level dependent noise.

F3

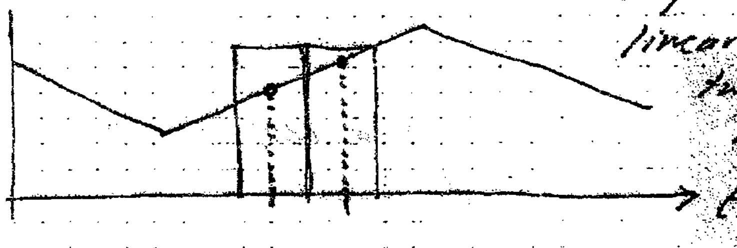


Figure 9: Piecewise linear signal, and two box splines measuring a linear piece.

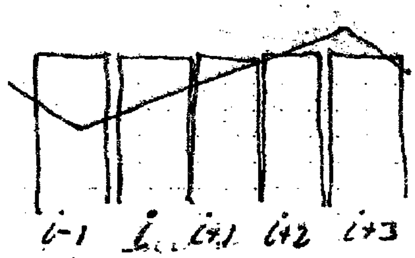


Figure 10: Piecewise linear signal, where discontinuities can be found.

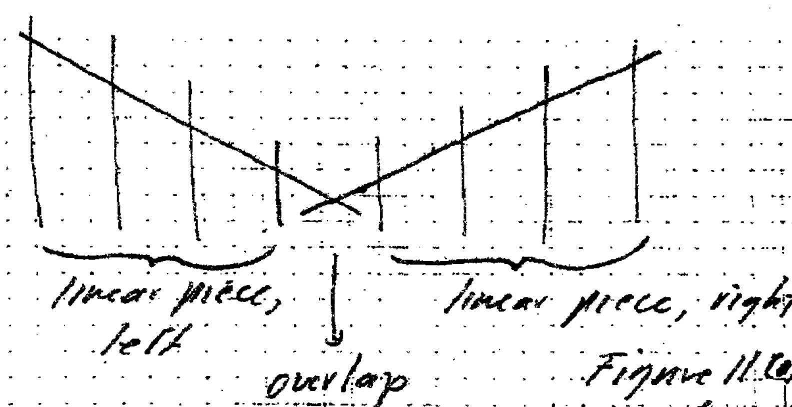


Figure 11(a) Finding the discontinuity location

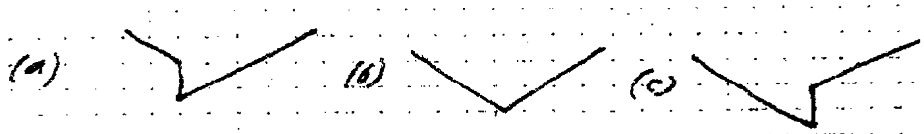


Figure 11(b): Finding the exact value

Fig.

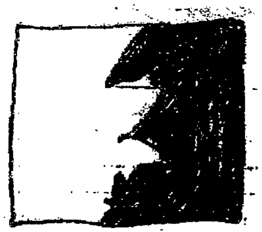


Figure 12(a) Arbitrary boundary

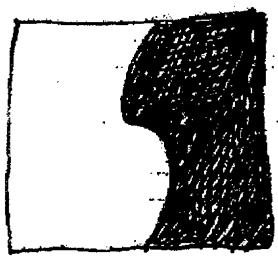


Figure 12(b) Smooth boundary

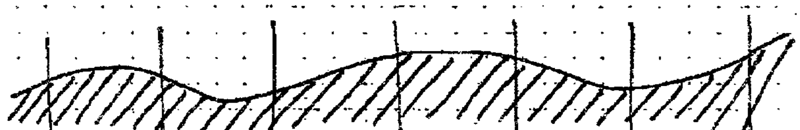


Figure 13: Bandlimited boundary function

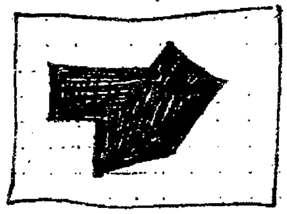


Figure 14: Piecewise linear boundary

F5

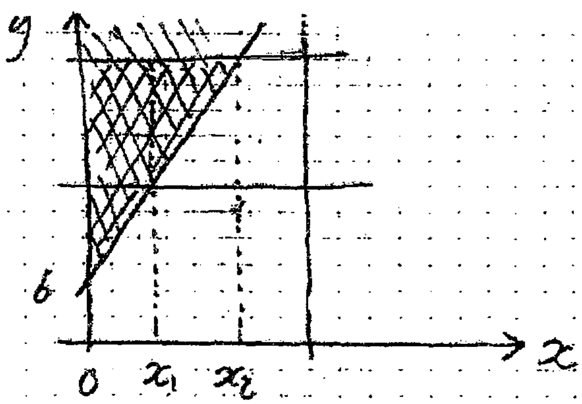


Figure 15: two-dimensional bilevel signal

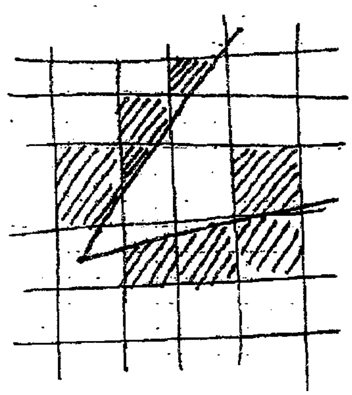


Figure 16: Intersection of boundaries

Appendix 2.A Annihilating filter method

The problem in spectral line analysis consists in estimating the frequencies of a sinusoidal signal from a set of values. The methods used for estimating such frequencies are known as high-resolution methods, for example MUSIC, ESPRIT and can be found in [67]. We define the following as the annihilating filter method:
Consider a signal $s[n], n \in \mathbb{Z}$ defined as a finite linear combination of K exponentials u_k^n,

$$s[n] = \sum_{k=0}^{K-1} c_k \, u_k^n \tag{2.84}$$

where c_k are real and u_k are real or complex valued. In the context of spectral line analysis $u_k = e^{i\omega_k}$ where ω_k is the kth frequency component of the signal $s[n]$.

Definition 2.3 *A filter $1 - z^{-1}u_k$ is called an annihilating filter for u_k^n if [67]*

$$(1 - z^{-1}u_k)u_k^n \equiv 0. \tag{2.85}$$

where z^{-1} is a shift or delay operator.

Suppose that $u_k \neq 0$, then $z = u_k$ satisfies (2.85) and is a zero of the filter $1 - z^{-1}u_k$. If there are K exponentials then there are K filters $1 - z^{-1}u_k$ each annihilating their respective u_k^n. This implies that the product of these filters

$$H(z) = \prod_{l=0}^{K-1} (1 - z^{-1}u_l) \tag{2.86}$$

for sure annihilates each exponentials, u_k^n, thus $H(z)$ is an annihilating filter of $s[n]$,

$$s[n] \cdot H(z) \equiv 0. \tag{2.87}$$

Therefore to find the values u_k we need to find the filter coefficients h_k in

$$H(z) = \sum_{m=0}^{K} h_k \, z^{-k} \tag{2.88}$$

such that (2.87) is satisfied. Substituting $H(z)$ defined in (2.88) in (2.87) we obtain

$$\sum_{k=0}^{K} h_k \, s[n] \, z^{-k} = 0, \quad n \in \mathbb{Z} \tag{2.89}$$

which equivalent to the following recurrence equation

$$\sum_{k=0}^{K} h_k \, s[n-k] = 0 \tag{2.90}$$

$$\mathbf{h} * \mathbf{s} = 0. \tag{2.91}$$

4